A fast Monte Carlo algorithm for the homogeneous set sandwich problem

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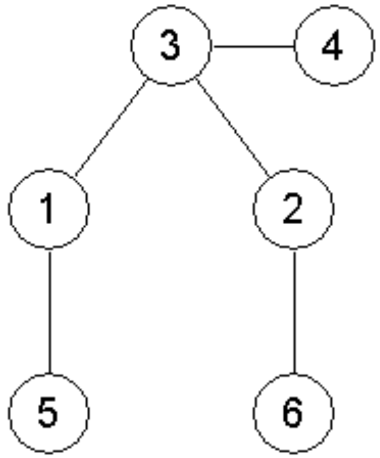
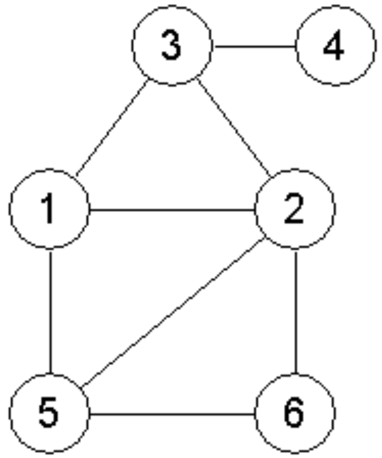
Sandwich Graphs

A graph $G_2(V_2,E_2)$ is said to be a *supergraph* of a graph $G_1(V_1,E_1)$ if and only if $V_2 = V_1$ and $E_2 \subseteq E_1$.

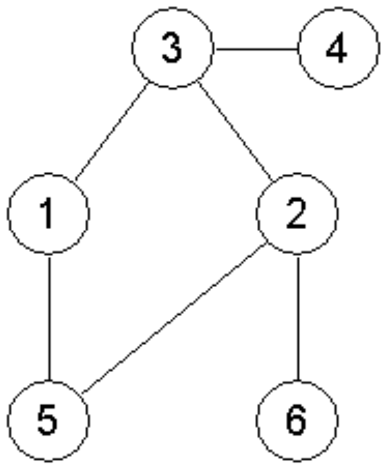


G_1



G_2
(supergraph of G_1)

A graph $G_S(V_S,E_S)$ is said to be a *sandwich graph* of a pair $G_1(V,E_1)$, $G_2(V,E_2)$ if and only if $V_S = V$ and $E_1 \subseteq E_S \subseteq E_2$.

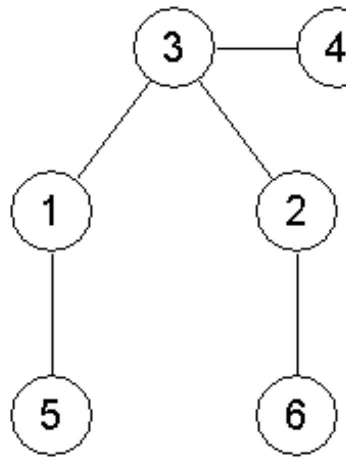


G_S
(sandwich graph of G_1,G_2)

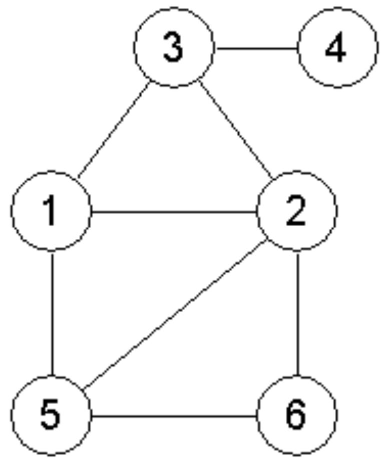
Sandwich Problems

Input: two graphs $G_1(V,E_1)$ and $G_2(V,E_2)$ such that G_2 is a *supergraph* of G_1.

Question: is there any sandwich graph G_S of pair (G_1,G_2) that has property $\prod$?



G_1



G_2
(supergraph of G_1)

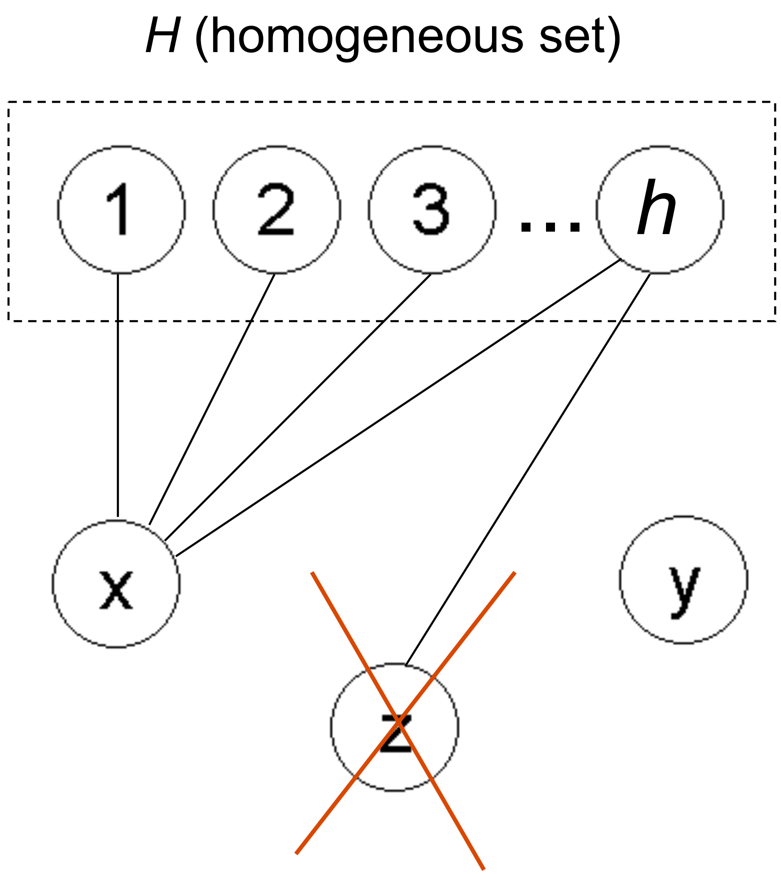
An edge (x,y) is said to be

- *mandatory*, if $(x,y) \in E_1$
- *forbidden*, if $(x,y) \notin E_2$
- *optional*, if $(x,y) \in E_2 \setminus E_1$

Homogeneous Sets

Let G (V, E) be a graph.

A set $H \subset V$ is a *homogeneous set* of G if and only if all vertices in H have exactly the same neighborhood outside H and $1 < |H| < |V|$.



The Homogeneous Set Sandwich Problem (HSSP): is there a sandwich graph of (G_1,G_2) which admits a homogeneous set?

History

$m = \min \{m_M, m_F\}$
$M = \max \{m_M, m_F\}$

AUTHORS	PUBLICATION	ALGORITHM		TIME	DET./RAND.
Cerioli, Everett, Figueiredo, Klein	IPL, 1998	1	Ehaustive Envelopment	$\cancel{O(n^4)}$ $\boxed{O(M\ n^2)}$	det.
~~Tang, Wang, Yeh~~	~~IPL, 2001~~	~~2~~	~~Strongly Connected Sinks~~	$\cancel{O(\Delta_2\ n^2)}$ $\boxed{O(m\ n)}$	~~det.~~
Figueiredo, P. de Sá	M.Sc., 2003 **+ IPL, 2004**	3	Two-Phase	$O(m\ M)$	det.
Figueiredo, Fonseca, P. de Sá, Spinrad	**LNCS, 2004 (WEA)**	4	Balanced Subsets	$O(n^{3,5})$	det.
		5	Monte Carlo	$O(n^3)$	rand.
Figueiredo, Fonseca, P. de Sá	**Algorithmica, 2005**	6	Harmonic Series	$O(n^3 \log n)$	det.
		7	Growing Cliques	$O(n^3 \log n)$	det.
		8	Las Vegas	$O(n^3)$	rand.
		9	Quick Fill	$O(n^3 \log m/n)$	det.
Bornstein, P. de Sá, Figueiredo	**IPL, 2006**	10	Pair Completion	$O(m\ n \log n)$	det.

QUALIFYING

Randomized Monte Carlo algorithms

- Give the correct answer with known probability *p*
- Their time complexity is computed deterministically

One-sided error Monte Carlo algorithms

<u>YES-biased Monte Carlo</u>

- Whenever the answer is YES, the answer is correct for certain (a certificate is given)

<u>NO-biased Monte Carlo</u>

Defined analogously

Several independent runs ⇔ any desired error ratio

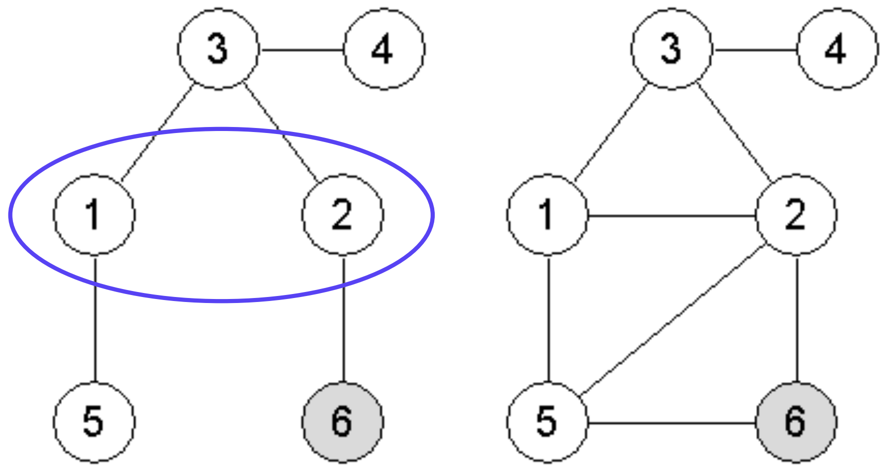
Bias vertices

Let $G_1(V,E_1)$, $G_2(V,E_2)$ be an input instance for the HSSP.

A vertex $b \in V$ is said to be a *bias vertex* of set $S \subseteq V \setminus \{ b \}$ iff there exists at least one mandatory edge $[b, x] \in E_1$ between b and some vertex $x \in S$ and, also, at least one forbidden edge $[b, y] \notin E_2$ between b and some vertex $y \in S$.

The set $B(S)$ containing all bias vertices of S is called its *bias set*.

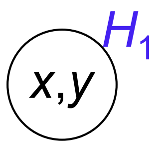
Sandwich instance (HSSP input)



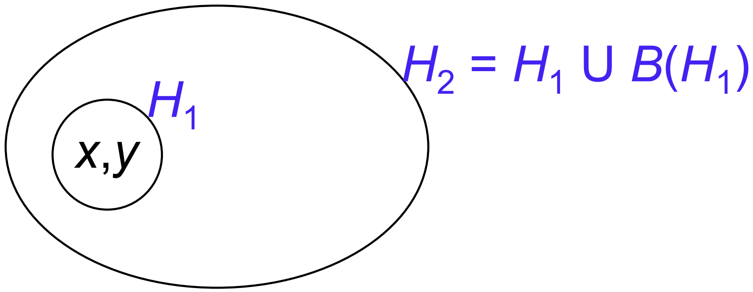
Sandwich Homogeneous Sets Characterization

A set $H \subset V$ is a sandwich homogeneous set of pair $G_1(V,E_1)$, $G_2(V,E_2)$ if and only if its bias set is the empty set.

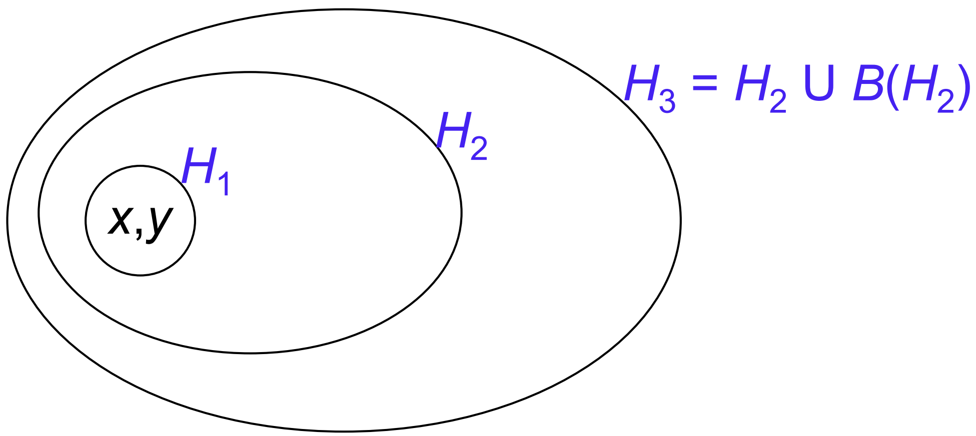
Bias Envelopment



Bias Envelopment



Bias Envelopment



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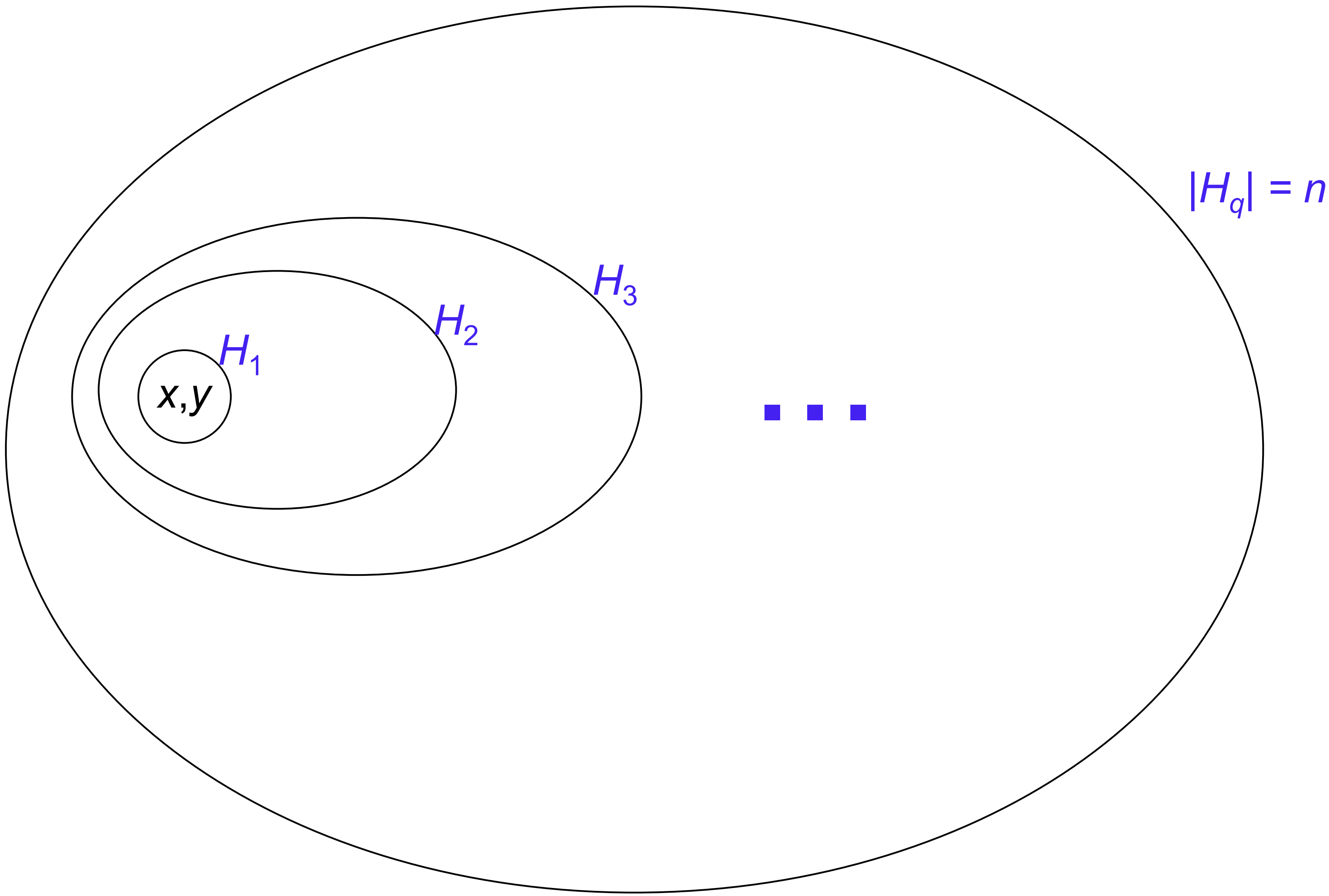

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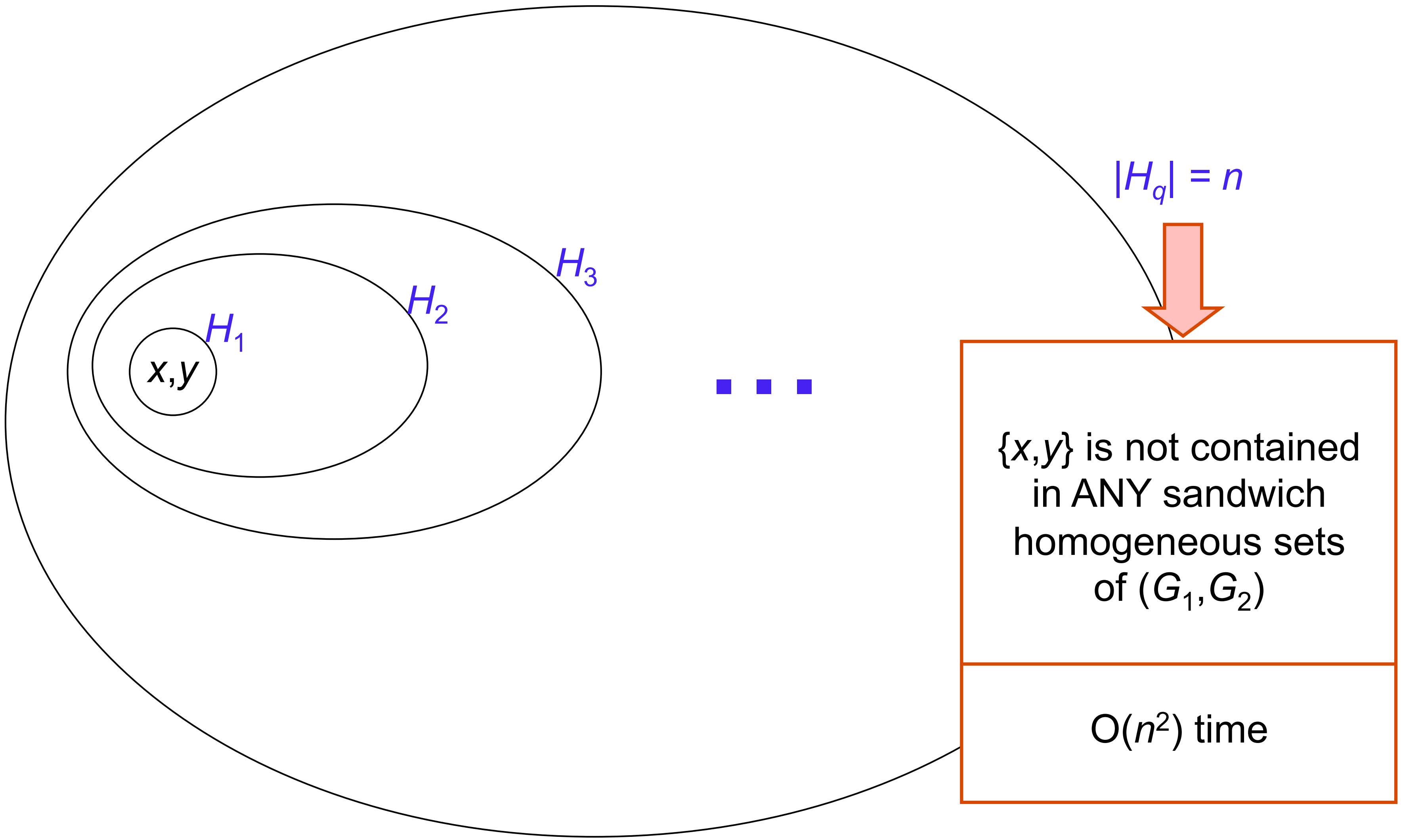


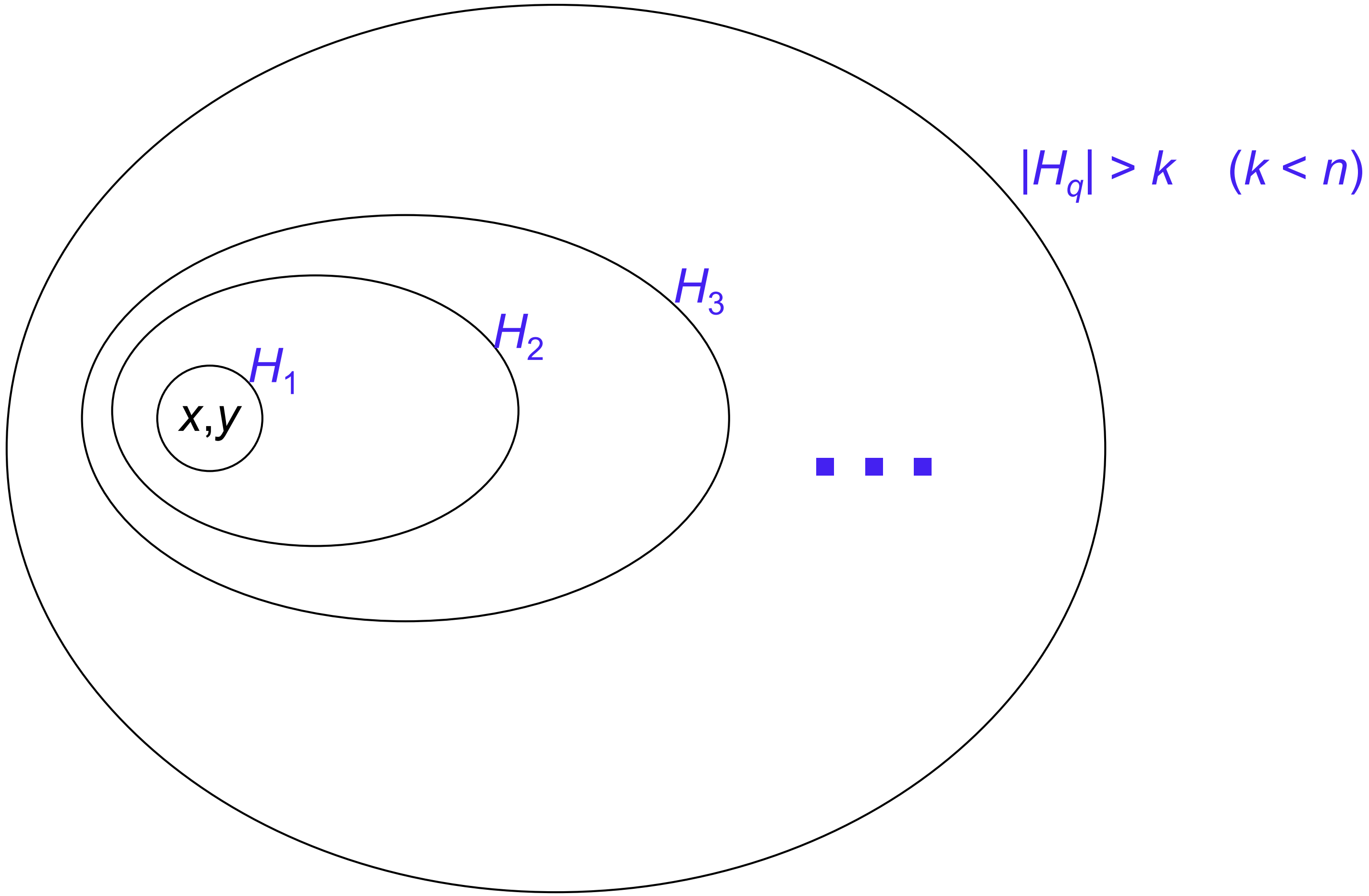
The Exhaustive Bias Envelopment algorithm

(CERIOLI, EVERETT, FIGUEIREDO, KLEIN, 1998)

1. For each pair of vertices $x, y \in V$ do
   1.1. $H \leftarrow \{x, y\}$
   1.2. While $|H| < n$ do
      1.2.1. Find the bias set T of H
      1.2.2. If $T = \emptyset$ then
         1.2.1.1 Return YES.
      1.2.3. $H \leftarrow H \cup T$
2. Return NO.

Bias Envelopment

Incomplete Bias Envelopment



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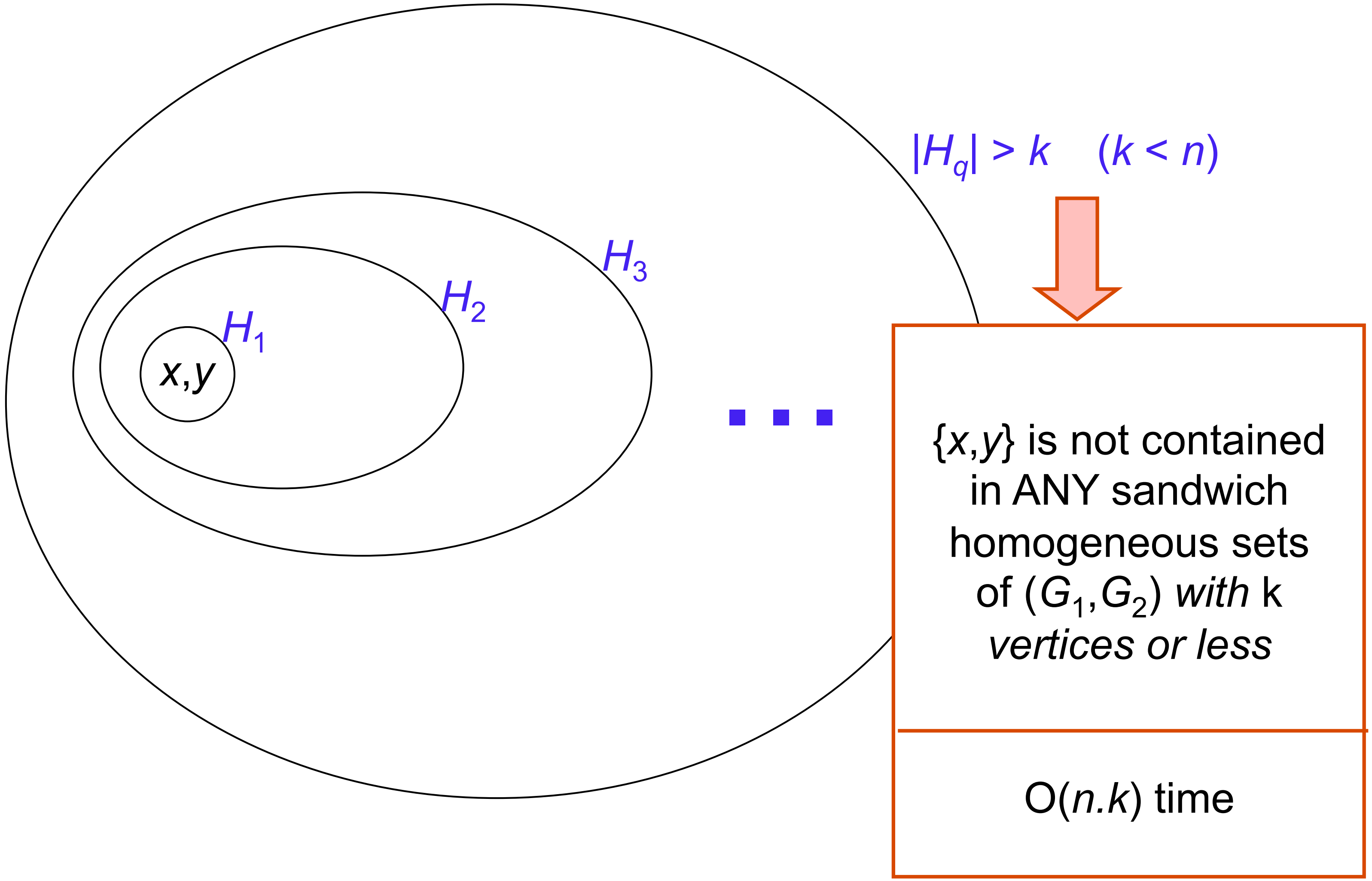
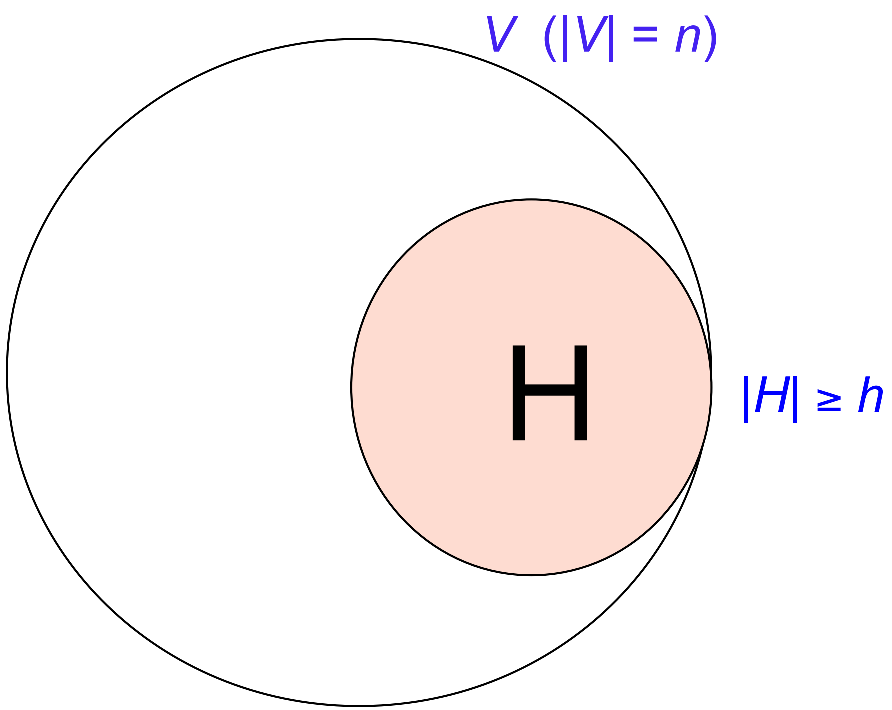


The Monte Carlo HSSP algorithm

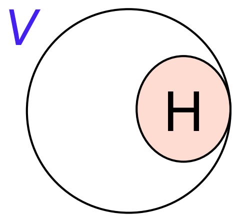
Let $G_1(V,E_1)$, $G_2(V,E_2)$ be an input instance for the HSSP.

Suppose there is a sandwich homogeneous set $H \subset V$ with h vertices or more.



The Monte Carlo HSSP algorithm

Hypothesis:



$|H| \geq h$

What is the probability $\overline{p_1}$ that a random pair of vertices $\{x,y\} \subset V$ is NOT contained in H ?

The Monte Carlo HSSP algorithm

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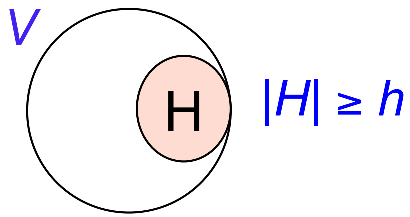

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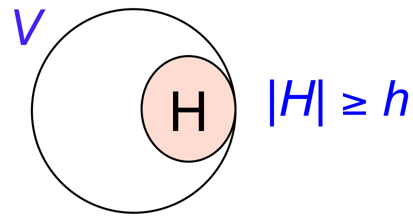
$$\overline{p_1} \leq 1 - \frac{h(h-1)}{n(n-1)}$$

The Monte Carlo HSSP algorithm

Hypothesis: 

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The Monte Carlo HSSP algorithm

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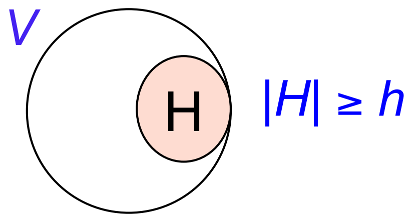

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$$\overline{p_t} \leq \left(1 - \frac{h(h-1)}{n(n-1)}\right)^t$$

The Monte Carlo HSSP algorithm

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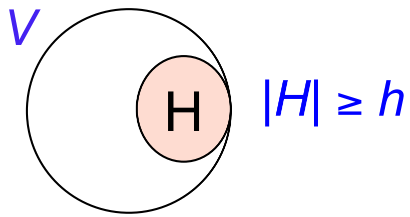

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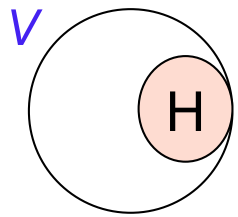
Now, what is the probability p_t that, running t Bias Envelopment procedures (starting from t random pairs of vertices), a sandwich homogeneous set is successfully found?

The Monte Carlo HSSP algorithm

Hypothesis: 

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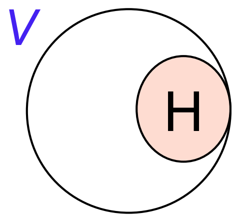
Hypothesis: 

$|H| \geq h$

$$p_t \geq 1 - \left(1 - \frac{h(h-1)}{n(n-1)}\right)^t$$

Fix $p_t \geq \boldsymbol{p} = 1 - \varepsilon$

$$h(t) = \left\lfloor \frac{1 + \sqrt{1 + 4(n^2 - n)(1 - (1-p)^{1/t})}}{2} \right\rfloor$$

Running the Bias Envelopment on *t* random pairs suffices to find a sandwich homogeneous set with probability at least ***p***, *in case there exists any with* h(t) *vertices or more*.

The Monte Carlo HSSP algorithm

But the algorithm is meant to find one, if there exists any, *no matter its size*.

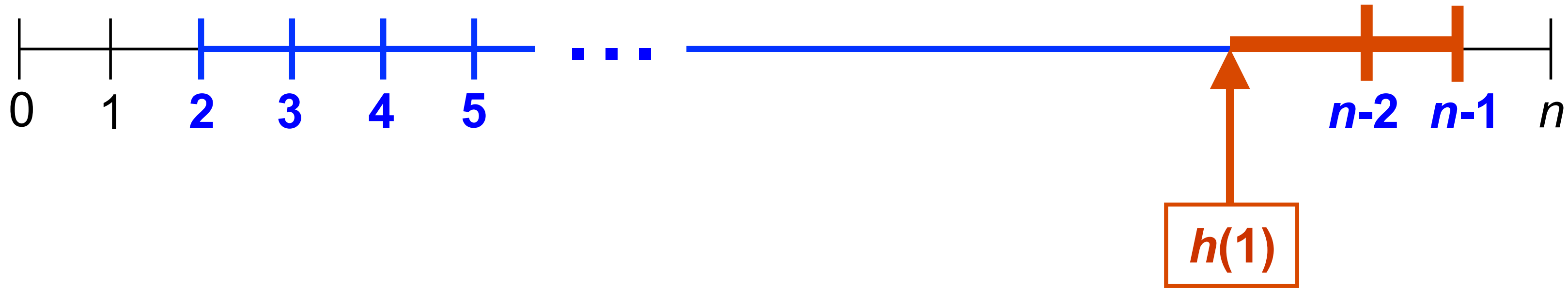
What is the number *t'* of Bias Envelopment procedures (on random pairs) that grants this?

$$h(t') = 2$$

The Monte Carlo HSSP algorithm

Number *t* of Bias Envelopment procedures undertaken on random pairs of vertices:	Minimum integer *h*(*t*) such that *t* Bias Envelopment executions (on random pairs) suffice to find some sandwich homogeneous set, *in case there exists any with* h(t) *vertices or more*:
0	we don't know anything

The Monte Carlo HSSP algorithm



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The Monte Carlo HSSP algorithm

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The Monte Carlo HSSP algorithm

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The Monte Carlo HSSP algorithm

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0	we don't know anything
1	*h*(1)
2	*h*(2)
...	...
t'	*h*(*t'*) = 2

The Monte Carlo HSSP algorithm

Determining *t'* ...

$$h(t) = \left\lfloor \frac{1 + \sqrt{1 + 4(n^2 - n)(1 - (1-p)^{1/t})}}{2} \right\rfloor$$, given a fixed $\boldsymbol{p} = 1 - \varepsilon$

$$t' = \frac{\ln(1-p)}{\ln\left(1 - \frac{2}{n(n-1)}\right)} = \left(\ln \frac{1}{\boldsymbol{\varepsilon}}\right) \Theta(n^2) \;=\; \Theta(n^2)$$

The Monte Carlo HSSP algorithm

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But this leads to an O(n^4) algorithm!!!!!!

The Monte Carlo HSSP algorithm

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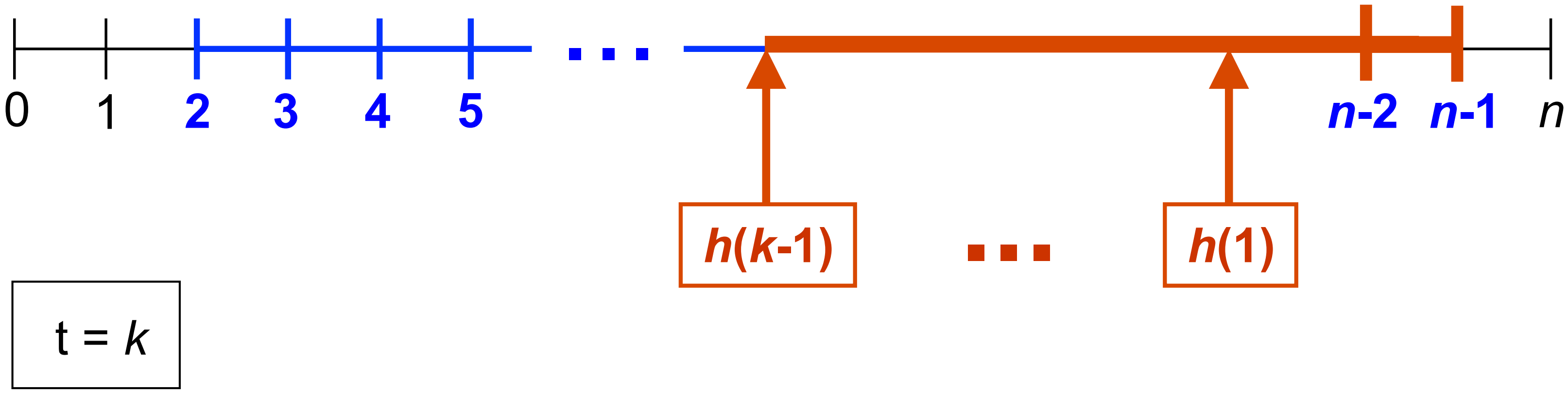
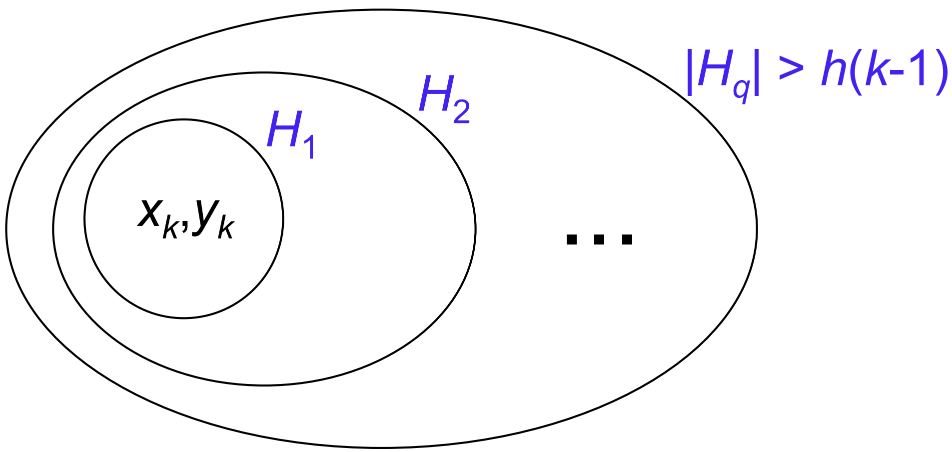
~~But this leads to an $O(n^4)$ algorithm!!!!!!~~

NO, IT DOESN'T.

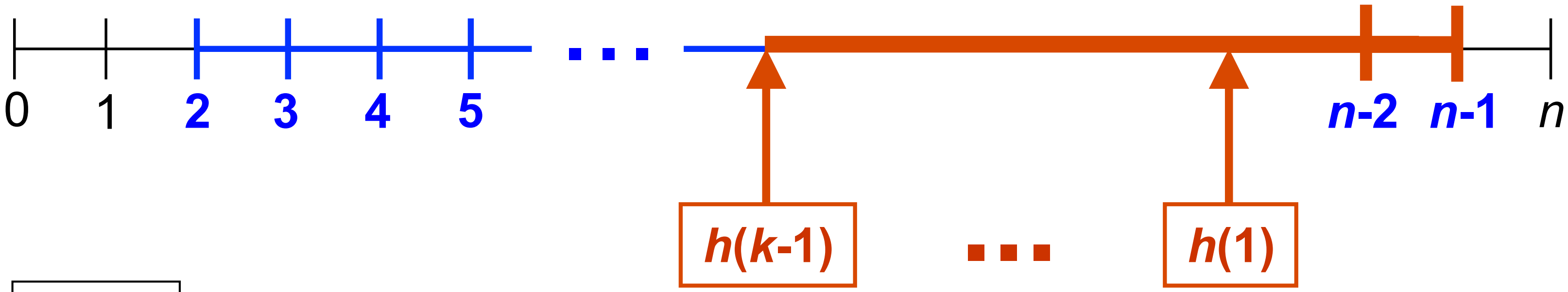
The Monte Carlo HSSP algorithm



- pick a random pair $\{x_k, y_k\}$
- Bias Envelopment



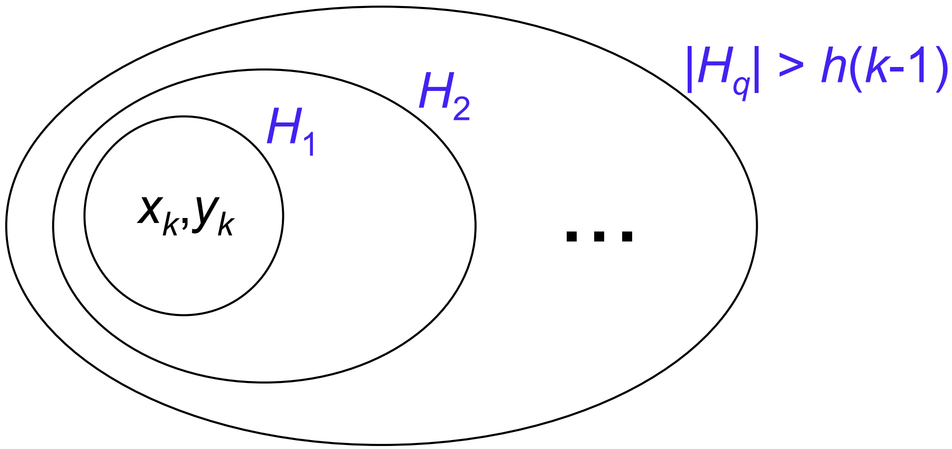
The Monte Carlo HSSP algorithm



t = k

Two possibilities:

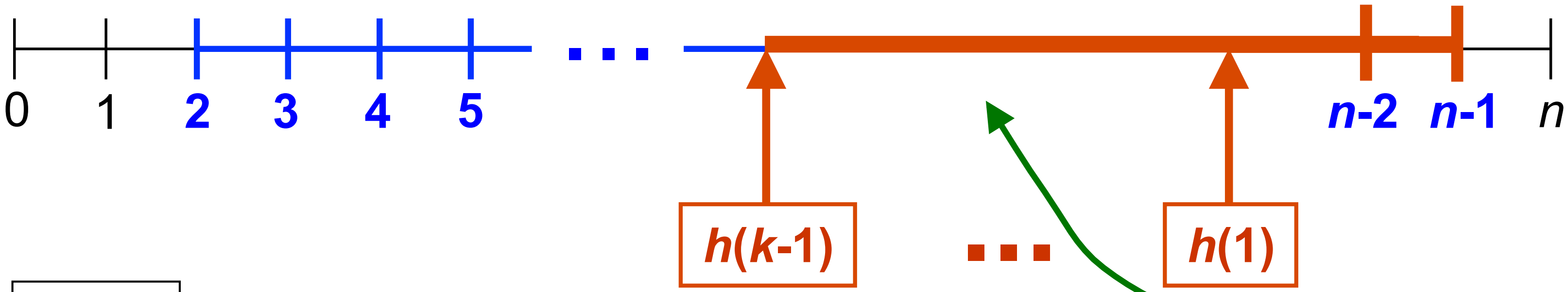
- pick a random pair $\{x_k, y_k\}$
- Bias Envelopment



(1) THERE IS a sandwich homogeneous set with more than $h(k\text{-}1)$ vertices

(2) THERE IS NO sandwich homogeneous set with more than $h(k\text{-}1)$ vertices

The Monte Carlo HSSP algorithm



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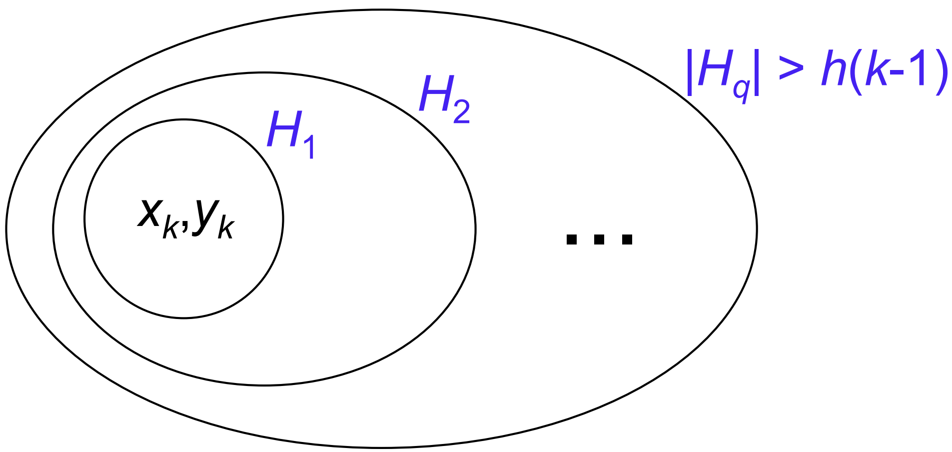

(1) THERE IS a sandwich homogeneous set with more than $h(k\text{-}1)$ vertices **OK!**

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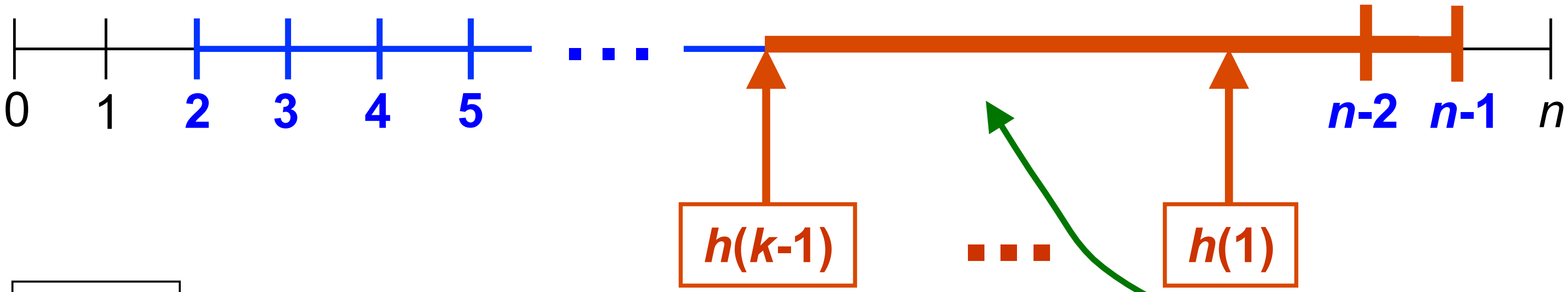
The Monte Carlo HSSP algorithm



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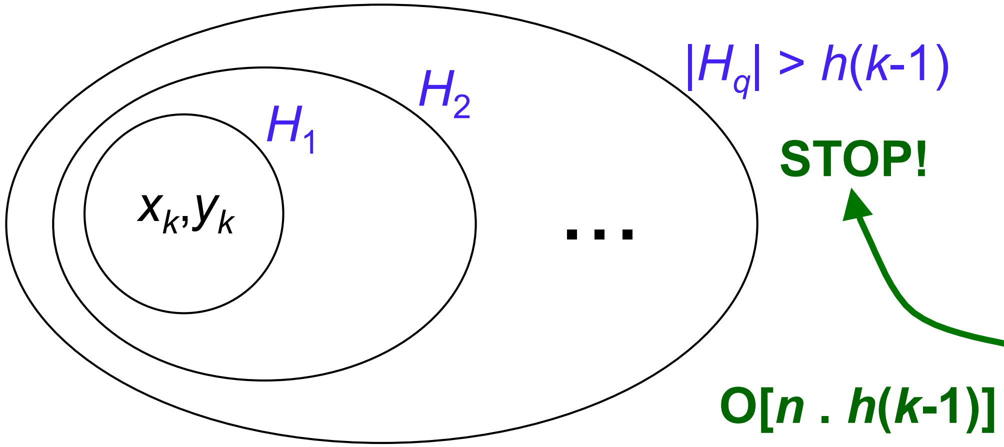

The Monte Carlo HSSP algorithm

1. $h \leftarrow n$
3. $t \leftarrow 0$
4. While $h > 2$ do
3.1. $t \leftarrow t + 1$
3.2. $(v_1, v_2) \leftarrow$ random pair of vertices
3.3. If *Incomplete Bias Envelopment* (v_1, v_2, h) = YES
3.3.1. Return YES.
3.4. $$h = \left\lfloor \frac{1 + \sqrt{1 + 4(n^2 - n)(1 - (1-p)^{1/t})}}{2} \right\rfloor$$
4. Return NO.

The Monte Carlo HSSP algorithm

Analysis:

$$\sum_{t=1}^{t'} O\left[nh(t)\right]$$

The Monte Carlo HSSP algorithm

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$$h(t) = \left\lfloor \frac{1+\sqrt{1+4(n^2-n)(1-(1-p)^{1/t})}}{2} \right\rfloor$$

$$h(t) \leq \Theta(n)\sqrt{\frac{1}{\Theta(t)}} = \frac{\Theta(n)}{\Theta(\sqrt{t})}$$

The Monte Carlo HSSP algorithm

Analysis:

$$\sum_{t=1}^{t'} O\big[nh(t)\big] = \sum_{t=1}^{\Theta(n^2)} \frac{O(n^2)}{O(\sqrt{t})}$$

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The Monte Carlo HSSP algorithm

Analysis:

$$\sum_{t=1}^{t'} O\left[nh(t)\right] = \sum_{t=1}^{\Theta(n^2)} \frac{O(n^2)}{O(\sqrt{t})} \quad = O(n^3)$$

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MUITO OBRIGADO.

THANKS.