Biased coins, blindfold players

Vinícius G. Pereira de Sá

based on the paper "Blind-friendly von Neumann's heads or tails",
to appear in The American Mathematical Monthly,
joint work with Celina M. H. de Figueiredo

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…back in 2007

Algoritmos Randomizados: Introdução

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Guilherme Fonseca
Manoel Lemos
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impa

26º Colóquio Brasileiro de Matemática
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Guilherme Dias da Fonseca

What's in this talk?

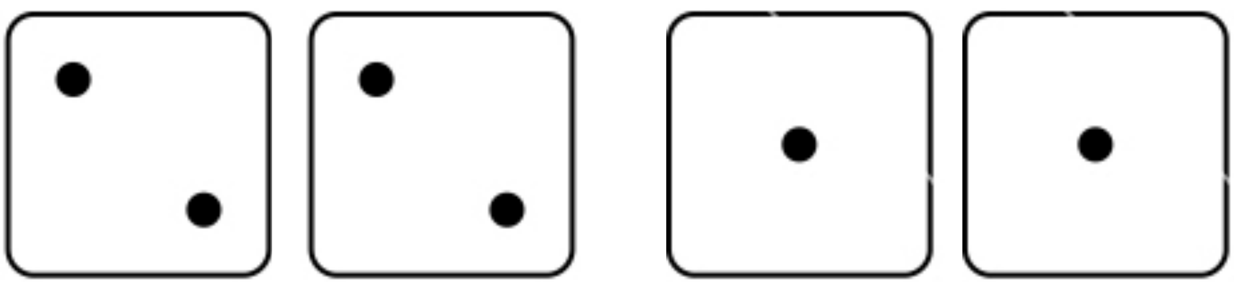
- biased randomness → unbiased randomness
- counterintuitive probabilities
  - a simple 2-player dice game (a triple or two straight doubles?)
  - conditioning on seemingly irrelevant knowledge
  - an interesting lemma
- fair heads or tails with a "concealed" biased coin
  - how *not* to do it
  - how to do it

a triple or two straight doubles?

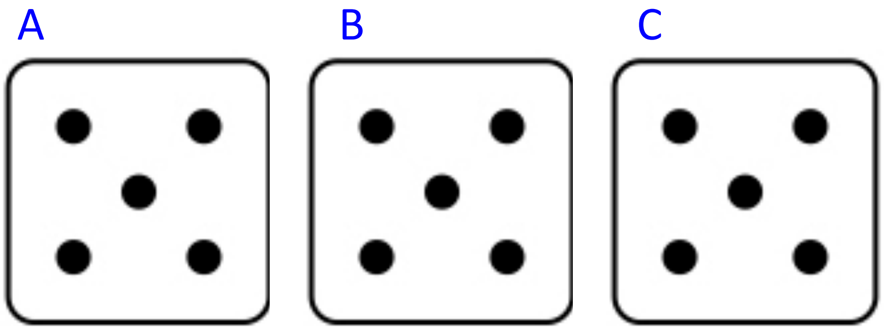
A triple or two straight doubles?

A B C

A B C D

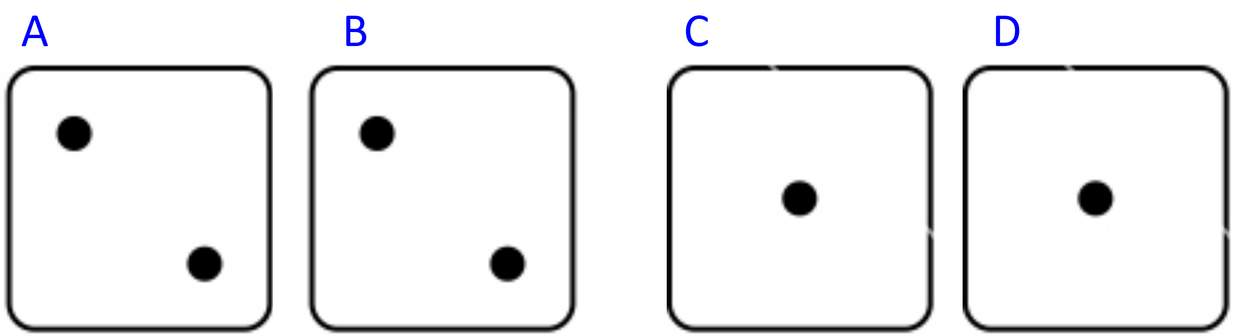


A triple or two straight doubles?



E_3 := A = B = C

$$\Pr\{E_3\} = \sum_{i \in \Omega} p_i^3$$



$E_{2,2}$:= A = B and C = D

$$\Pr\{E_{2,2}\} = \left(\sum_{i \in \Omega} p_i^2 \right)^2$$

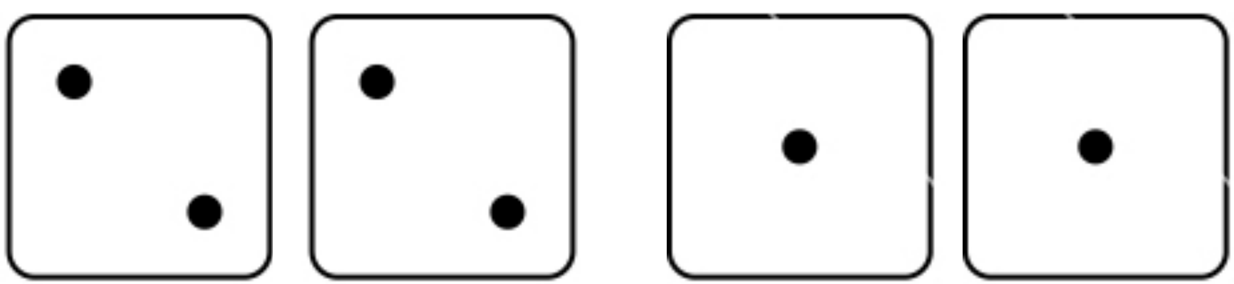
A triple or two straight doubles?

A B C

E_3 := A = B = C

$$\Pr\{E_3\} = \sum_{i\in\Omega} p_i^3$$

A B C D



$E_{2,2}$:= A = B and C = D

$$\Pr\{E_{2,2}\} = \left(\sum_{i\in\Omega} p_i^2\right)^2$$

A triple or two straight doubles?

$$\Pr\{E_3\} = \sum_{i \in \Omega} p_i^3$$

$$\Pr\{E_{2,2}\} = \left(\sum_{i \in \Omega} p_i^2 \right)^2$$

A triple or two straight doubles?

Cauchy inequality:

$$\left(\sum_{i\in\Omega} x_i y_i\right)^2 \leq \left(\sum_{i\in\Omega} x_i^2\right)\left(\sum_{i\in\Omega} y_i^2\right)$$

$$\Pr\{E_3\} = \sum_{i\in\Omega} p_i^3$$

$$\geq$$

$$\Pr\{E_{2,2}\} = \left(\sum_{i\in\Omega} p_i^2\right)^2$$

A triple or two straight doubles?

Cauchy inequality:

$$\left(\sum_{i\in\Omega} x_i y_i\right)^2 \leq \left(\sum_{i\in\Omega} x_i^2\right)\left(\sum_{i\in\Omega} y_i^2\right)$$

$$x_i = p_i^{3/2} \qquad y_i = p_i^{1/2}$$

$$\Pr\{E_3\} = \sum_{i\in\Omega} p_i^3$$

$$\geq$$

$$\Pr\{E_{2,2}\} = \left(\sum_{i\in\Omega} p_i^2\right)^2$$

A triple or two straight doubles?

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$$\left(\sum_{i\in\Omega} p_i^2\right)^2 \leq \left(\sum_{i\in\Omega} p_i^3\right)\cancelto{1}{\left(\sum_{i\in\Omega} p_i\right)} = \sum_{i\in\Omega} p_i^3$$

$$\Pr\{E_3\} = \sum_{i\in\Omega} p_i^3$$

$$\geq$$

$$\Pr\{E_{2,2}\} = \left(\sum_{i\in\Omega} p_i^2\right)^2$$

A triple or two straight doubles?

$$\left(\sum_{i\in\Omega} p_i^2\right)^2 = \boxed{\Pr\{E_{2,2}\} \ \leq \ \Pr\{E_3\}} = \sum_{i\in\Omega} p_i^3$$

- perfectly fair dice
- perfectly unfair (loaded) dice
- coins

A triple or two straight doubles?

$$\left(\sum_{i\in\Omega} p_i^2\right)^2 = \boxed{\Pr\{E_{2,2}\} \ \le \ \Pr\{E_3\}} = \sum_{i\in\Omega} p_i^3$$

- perfectly fair dice

$p_i = 1/n$ for all $i \in \Omega$ $\Rightarrow$ $\Pr\{E_3\} = \Pr\{E_{2,2}\} = \dfrac{1}{n^2}$

$\boxed{n = |\,\Omega\,|}$

- perfectly unfair (loaded) dice

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A triple or two straight doubles?

$$\left(\sum_{i\in\Omega} p_i^2\right)^2 = \boxed{\Pr\{E_{2,2}\} \ \leq \ \Pr\{E_3\}} = \sum_{i\in\Omega} p_i^3$$

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  $p_1 = 1$

  $p_i = 0$ for $i \in \{2, \ldots, n\}$ $\Rightarrow$ $\Pr\{E_3\} = \Pr\{E_{2,2}\} = 1$

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A triple or two straight doubles?

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$\boxed{n = |\,\Omega\,|}$

$$\begin{aligned} p_1 &= 1 \\ p_i &= 0 \ \text{ for } i \in \{2, \ldots, n\} \end{aligned} \quad \Rightarrow \quad \Pr\{E_3\} = \Pr\{E_{2,2}\} = 1$$

- coins

$$\begin{aligned} p_1 &= p \\ p_2 &= 1 - p \end{aligned} \quad \Rightarrow \quad \Pr\{E_3\} - \Pr\{E_{2,2}\} =$$

$$= \quad \left[p^3 + (1-p)^3\right] - \left[p^2 + (1-p)^2\right]^2 \ =$$

$$= \quad -4p^4 + 8p^3 - 5p^2 + p$$

A triple or two straight doubles?

- coins

$$p_1 = p \qquad p_2 = 1 - p$$

$$\Rightarrow \Pr\{E_3\} - \Pr\{E_{2,2}\} =$$

$$= \left[p^3 + (1-p)^3\right] - \left[p^2 + (1-p)^2\right]^2 =$$

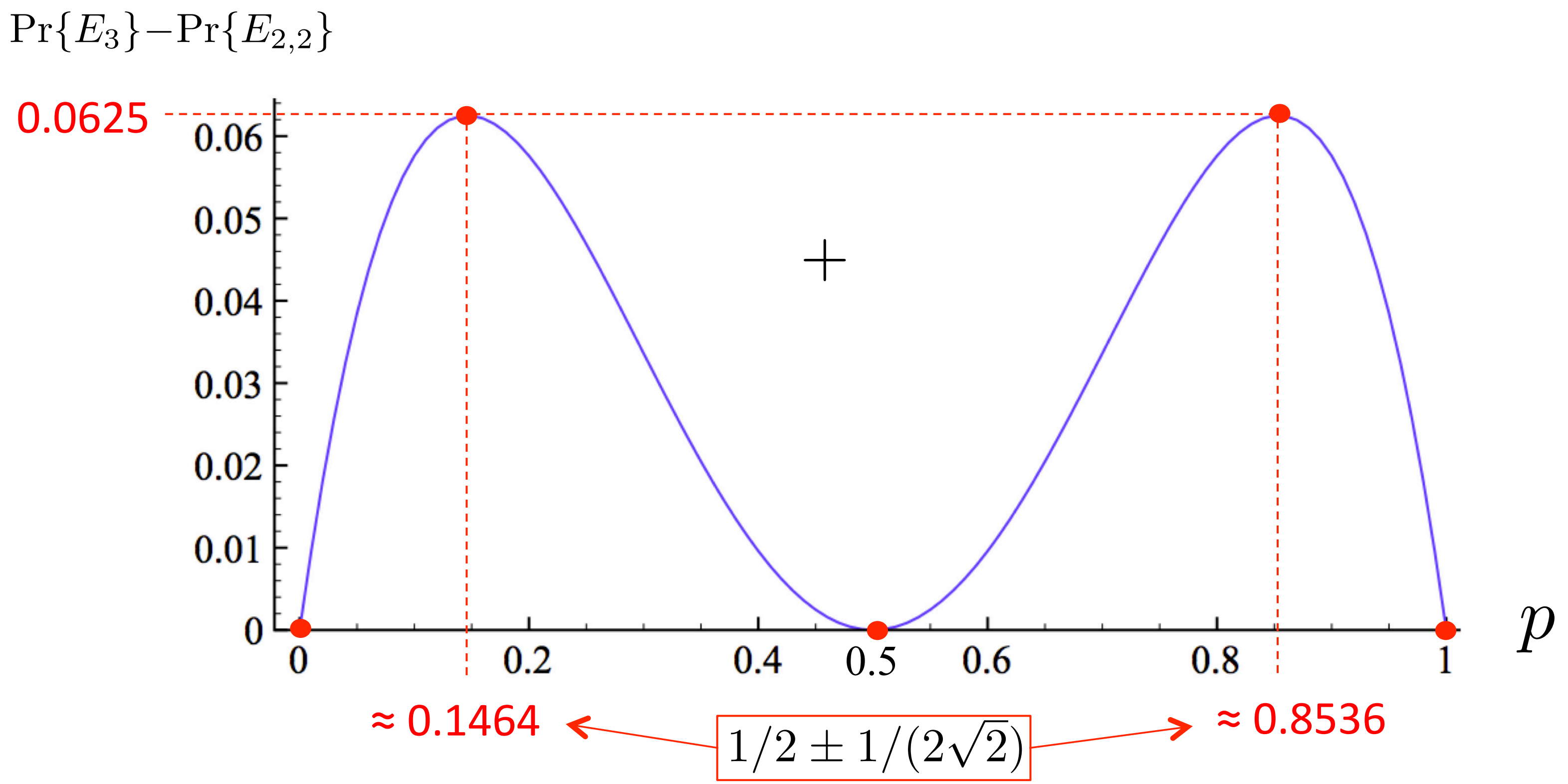
$$= -4p^4 + 8p^3 - 5p^2 + p$$

A triple or two straight doubles?

- coins

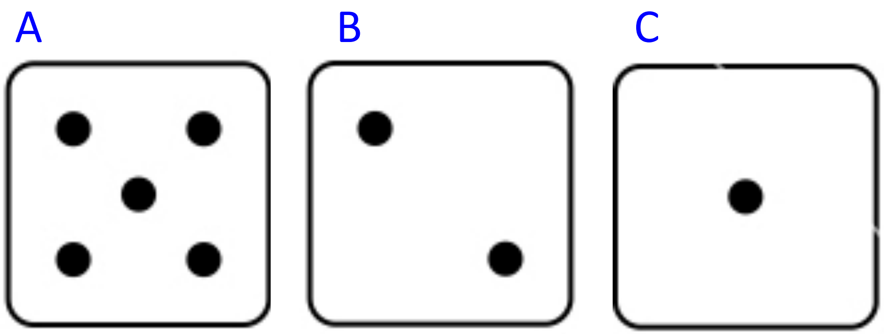
$$p_1 = p$$
$$p_2 = 1 - p$$

$$\Pr\{E_3\} - \Pr\{E_{2,2}\} =$$
$$= \left[p^3 + (1-p)^3\right] - \left[p^2 + (1-p)^2\right]^2 =$$
$$= -4p^4 + 8p^3 - 5p^2 + p$$



seemingly irrelevant knowledge

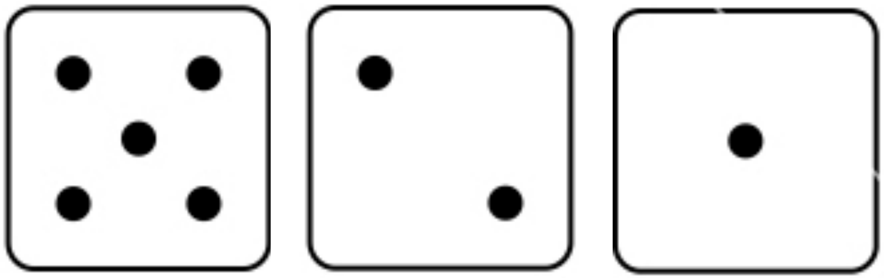
Seemingly irrelevant knowledge



A, B, C **independent** identically distributed (iid) random variables

Seemingly irrelevant knowledge

A B C



A, B, C **independent** identically distributed (iid) random variables

$$\Pr\{ C = B \} \overset{?}{=} \Pr\{ C = B \mid B \neq A \}$$

Seemingly irrelevant knowledge

$$\Pr\{ C = B \} \stackrel{?}{=} \Pr\{ C = B \mid B \neq A \}$$

Seemingly irrelevant knowledge

Pr{ C = B } $\overset{?}{=}$ Pr{ C = B | B ≠ A }

$$\Pr\{C = B\} = \sum_{i \in \Omega} p_i^2$$

$$\Pr\{C = B \mid B \neq A\} = \frac{\Pr\{C = B \neq A\}}{\Pr\{B \neq A\}} = \frac{\sum_{i \in \Omega}[p_i^2(1 - p_i)]}{\sum_{i \in \Omega}[p_i(1 - p_i)]}$$

Seemingly irrelevant knowledge

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$$\Pr\{C = B\} = \sum_{i \in \Omega} p_i^2$$

$$\neq$$

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Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \;\leq\; \Pr\{C = B\}$$

$$\Pr\{C = B\}$$

$$\rotatebox{-90}{$\geq$}$$

$$\Pr\{C = B \mid B \neq A\}$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \ \leq \ \Pr\{C = B\}$$

$$\frac{\Pr\{C = B \neq A\}}{1 - \Pr\{B = A\}} \ \leq \ \Pr\{C = B\}$$

$$\Pr\{C = B\}$$

$$\Pr\{C = B \mid B \neq A\}$$

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$$\begin{aligned} \Pr\{C = B \neq A\} \;\leq\;& \Pr\{C = B\} \\ & - \Pr\{C = B\}\ \Pr\{B = A\} \end{aligned}$$

$$\boxed{\begin{array}{c} \Pr\{C = B\} \\ \geq \\ \Pr\{C = B \mid B \neq A\} \end{array}}$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \;\leq\; \Pr\{C = B\}$$

$$\frac{\Pr\{C = B \neq A\}}{1 - \Pr\{B = A\}} \;\leq\; \Pr\{C = B\}$$

$$\Pr\{C = B \neq A\} \;\leq\; \Pr\{C = B\} - \Pr\{C = B\}\ \Pr\{B = A\}$$

A, B, C iid

$$\Pr\{B = A\} = \Pr\{C = B\}$$

$$\Pr\{C = B \neq A\} \;\leq\; \Pr\{C = B\} - \Pr\{C = B\}^2$$

$$\Pr\{C = B\}$$

$$\geq$$

$$\Pr\{C = B \mid B \neq A\}$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \;\leq\; \Pr\{C = B\}$$

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A, B, C iid

$$\Pr\{B = A\} = \Pr\{C = B\}$$

$$\Pr\{C = B\}$$

$$\Pr\{C = B \mid B \neq A\}$$

$$\Pr\{C = B \neq A\} \;\leq\; \Pr\{C = B\} - \Pr\{C = B\}^2$$

$$\Pr\{C = B \neq A\} \;\leq\; \Pr\{C = B \neq A\} + \Pr\{C = B = A\} - \Pr\{C = B\}^2$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \;\leq\; \Pr\{C = B\}$$

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A, B, C iid

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$$\Pr\{C = B\}$$

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$$\Pr\{C = B\}^2 = \left(\sum_{i \in \Omega} p_i^2\right)^2 \qquad \Pr\{C = B = A\} = \sum_{i \in \Omega} p_i^3$$

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A, B, C iid

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$$\Pr\{C = B\}^2 \leq \Pr\{C = B = A\}$$

$$\left(\sum_{i\in\Omega} p_i^2\right)^2 \;\leq\; \sum_{i\in\Omega} p_i^3$$

Seemingly irrelevant knowledge

$$\frac{\sum_{i\in\Omega}[p_i^2(1-p_i)]}{\sum_{i\in\Omega}[p_i(1-p_i)]} = \boxed{\Pr\{C=B \mid B\neq A\} \leq \Pr\{C=B\}} = \sum_{i\in\Omega} p_i^2$$

Example:

$$\Omega = \{1,2,3\}$$

$$p_1 = 0.8$$

$$p_2 = p_3 = 0.1$$

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$$\Omega = \{1,2,3\}$$
$$p_1 = 0.8$$
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$$\Pr\{C=B\} = 0.66$$

$$\Pr\{C=B \mid B\neq A\} \approx 0.429$$

Seemingly irrelevant knowledge

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- perfectly fair dice
- perfectly unfair (loaded) dice
- coins

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$p_i = 1/n \text{ for all } i \in \Omega$ ⇒ $\Pr\{C=B \mid B\neq A\} =$

$= \Pr\{C=B\} = 1/n$

$\boxed{n = |\Omega|}$

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  $p_i = 1/n$ for all $i \in \Omega$ ⇨ $\Pr\{C = B \mid B \neq A\} =$
  $= \Pr\{C = B\} = 1/n$

  $\boxed{n = |\,\Omega\,|}$

- perfectly unfair (loaded) dice

  $p_1 = 1$
  $p_i = 0$ for $i \in \{2, \ldots, n\}$ ⇨ $B \neq A$
  impossible event

- coins

Seemingly irrelevant knowledge

$$\frac{\sum_{i\in\Omega}[p_i^2(1-p_i)]}{\sum_{i\in\Omega}[p_i(1-p_i)]} = \boxed{\Pr\{C=B \mid B\neq A\} \leq \Pr\{C=B\}} = \sum_{i\in\Omega} p_i^2$$

- perfectly fair dice

  $p_i = 1/n \text{ for all } i \in \Omega \Rightarrow \Pr\{C=B \mid B\neq A\} = = \Pr\{C=B\} = 1/n$

  $\boxed{n = |\Omega|}$

- perfectly unfair (loaded) dice

  $p_1 = 1$
  $p_i = 0 \text{ for } i \in \{2,\ldots,n\} \Rightarrow B \neq A$

  impossible event

- coins

  $\Pr\{C=B\} = p_1^2 + p_2^2 = p^2 + (1-p)^2 = 2p^2 - 2p + 1$ minimum at (0.5, 0.5)

Seemingly irrelevant knowledge

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  $p_i = 1/n \text{ for all } i \in \Omega$ $\Rightarrow$ $\Pr\{C=B \mid B\neq A\} = \Pr\{C=B\} = 1/n$

  $\boxed{n = |\Omega|}$

- perfectly unfair (loaded) dice

  $p_1 = 1$

  $p_i = 0 \text{ for } i \in \{2,\ldots,n\}$ $\Rightarrow$ $B \neq A$

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  $$\Pr\{C=B \mid B\neq A\} = \frac{p^2(1-p)+(1-p)^2p}{2p-2p^2} = \frac{p-p^2}{2p-2p^2}$$

Seemingly irrelevant knowledge

$$\frac{\sum_{i\in\Omega}[p_i^2(1-p_i)]}{\sum_{i\in\Omega}[p_i(1-p_i)]} = \boxed{\Pr\{C=B \mid B\neq A\} \leq \Pr\{C=B\}} = \sum_{i\in\Omega} p_i^2$$

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  $$\Pr\{C=B \mid B\neq A\} = \frac{p^2(1-p)+(1-p)^2p}{2p-2p^2} = \frac{p-p^2}{2p-2p^2} = \mathbf{0.5}$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \quad = \quad \frac{p^2(1-p) + (1-p)^2 p}{2p - 2p^2} \quad = \quad \frac{p - p^2}{2p - 2p^2} \quad = \quad \mathbf{0.5}$$

Seemingly irrelevant knowledge

$$\Pr\{C = B \mid B \neq A\} \qquad = \mathbf{0.5}$$

Lemma:

Given three independent Bernoulli random variables A, B, and C with success probability $0 < p < 1$, we have

$$\Pr\{C = B \mid B \neq A\} \;=\; \mathbf{0.5} \qquad \text{regardless of } p.$$

Lemma:

Given three independent Bernoulli random variables A, B, and C with success probability $0 < p < 1$, we have

$$\Pr\{C = B \mid B \neq A\} = \mathbf{0.5} \quad \text{regardless of } p.$$

fair heads or tails
with a concealed biased coin

von Neumann's idea

take a biased coin, flip it twice

(H)eads-(T)ails, Player 1 wins

T-H, Player 2 wins

H-H or T-T, start over

$\boxed{p = \Pr\{\mathrm{H}\}}$

$$\Pr\{\text{H-T}\} = \Pr\{\text{T-H}\} = p\,(1-p)$$

von Neumann's idea

$\Pr\{\text{H-T}\} = \Pr\{\text{T-H}\} = p\,(1 - p)$

$\Pr\{\text{someone wins at a given turn}\} = \Pr\{\text{T-H}\} + \Pr\{\text{H-T}\} = 2\,p\,(1 - p)$

$$\text{expected \# flips until someone wins} = 2 \times \frac{1}{2\,p\,(1-p)} = \frac{1}{p\,(1-p)}$$

von Neumann's idea

$\Pr\{\text{H-T}\} = \Pr\{\text{T-H}\} = p\,(1-p)$

$\Pr\{\text{someone wins at a given turn}\} = \Pr\{\text{T-H}\} + \Pr\{\text{H-T}\} = 2\,p\,(1-p)$

$$\text{expected \# flips until someone wins} = 2 \times \frac{1}{2\,p\,(1-p)} = \frac{1}{p\,(1-p)}$$

many variations/improvements ever since, e.g.

P. Elias, The efficient construction of an unbiased random sequence, *The Annals of Mathematical Statistics* **43** (1972) 865–870.

Q. F. Stout, B. Warren, Tree algorithms for unbiased coin tossing with a biased coin, *The Annals of Probability* **12** (1984) 212–222.

S. Vembu, S. Verdú, Generating random bits from an arbitrary source: fundamental limits, *IEEE Transactions on Information Theory* **41** (1995) 1322–1332.

A. Srinivasan, D. Zuckerman, Computing with very weak random sources, *SIAM Journal on Computing* **28** (1999) 1433–1459.

Concealed biased coins

- a coin that is possibly biased (the exact bias is unknown)
- the players are not able to know whether the coin flip resulted H or T
- it is possible to infer a (mis)match between the two latest results

Concealed biased coins

- a coin that is possibly biased (the exact bias is unknown)
- the players are not able to know whether the coin flip resulted H or T
- it is possible to infer a (mis)match between the two latest results

a hand clap (Cl)

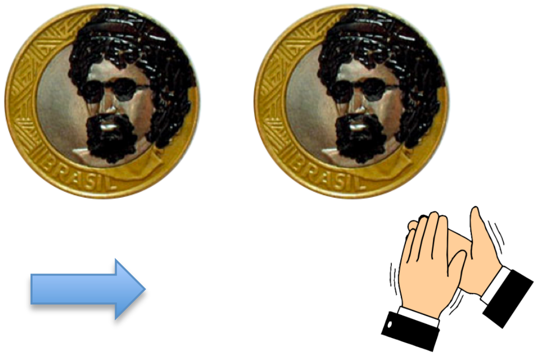
the latest result is **the same** as the previous

a whistle (W)

the latest result is **different** from the previous

Hand claps and whistles

Hand claps and whistles



Hand claps and whistles

Hand claps and whistles

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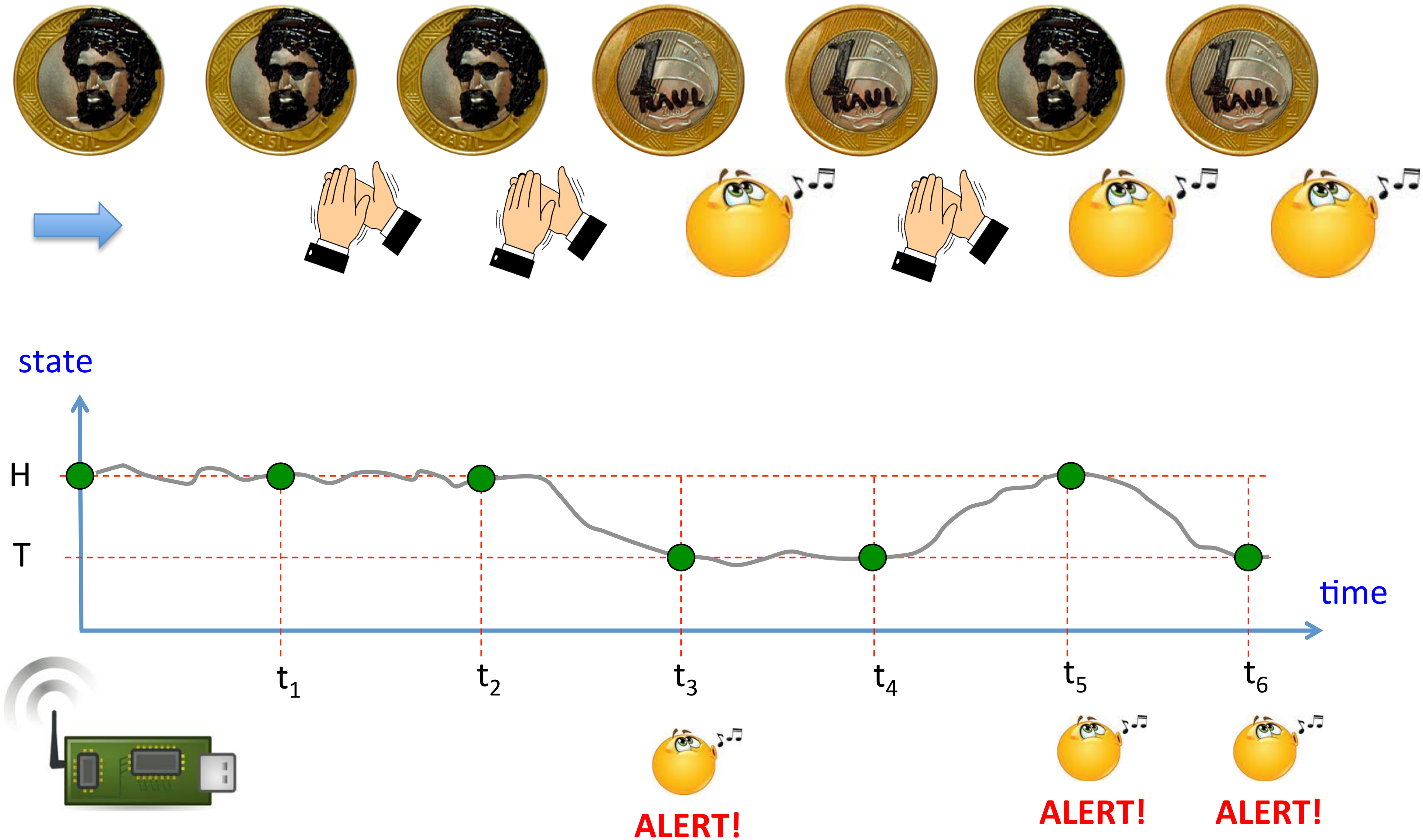
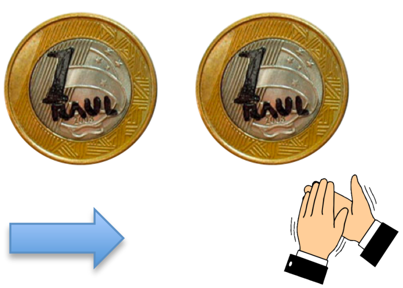


First attempt: Cl vs. W (2 flips)

Player 1: Cl

or



Player 2: W

or

$\boxed{p = \Pr\{\mathrm{H}\}}$

$$P_1 \;=\; \Pr\{\text{Player 1 wins}\} \;=\; p^2 + (1-p)^2 \;=\; 2p^2 - 2p + 1$$

First attempt: Cl vs. W (2 flips)

Player 1: Cl

Player 2: W

$$\boxed{p = \Pr\{\mathrm{H}\}}$$

$$P_1 \;=\; \Pr\{\text{Player 1 wins}\} \;=\; p^2 + (1-p)^2 \;=\; 2p^2 - 2p + 1$$

Second attempt: Cl-W vs. W-Cl (initial + 2 flips per turn)

Player 1: Cl-W

…………H | H | T

Cl W

or

…………T | T | H

Cl W

Player 2: W-Cl

…………T | H | H

W Cl

or

…………H | T | T

W Cl

From now on…

H

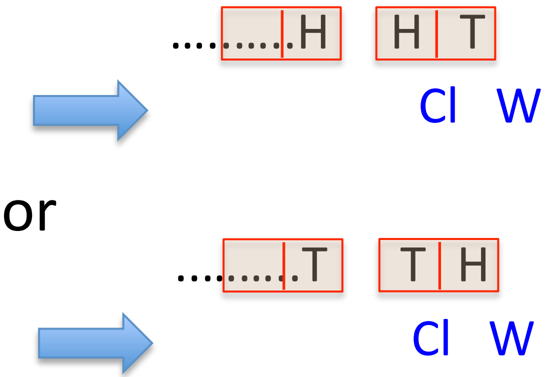
T

Cl

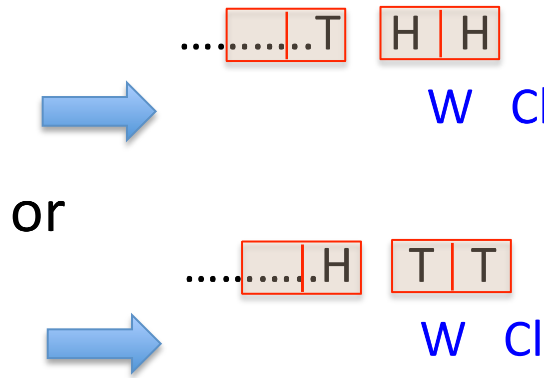
W

Second attempt: Cl-W vs. W-Cl (initial + 2 flips per turn)

Player 1: Cl-W



Player 2: W-Cl



P_1^H := probability that Player 1 wins the game provided the first coin flip yields a H

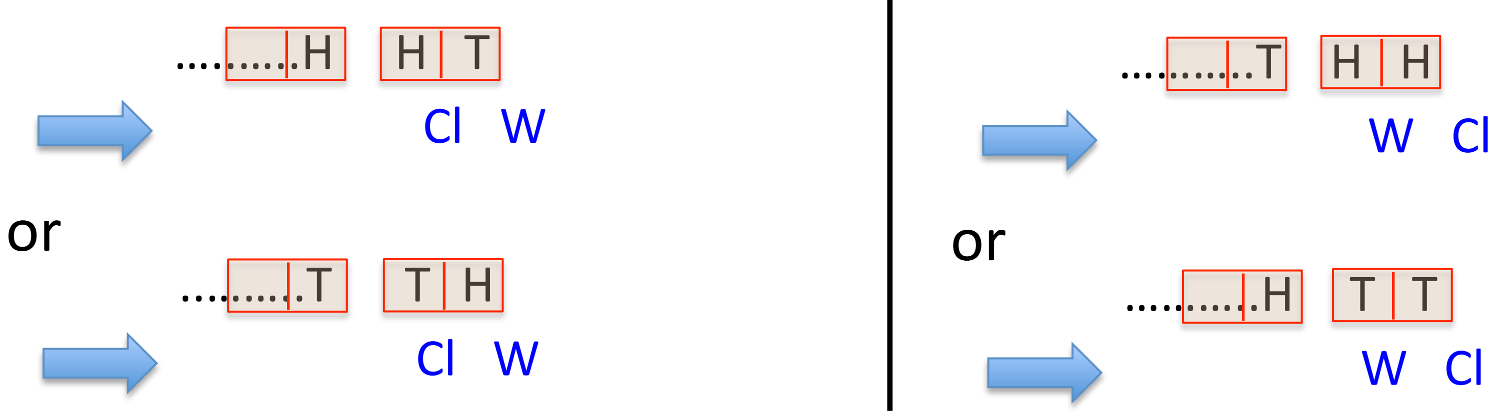
$$P_1 = P_1^H \cdot p + P_1^T \cdot (1-p)$$

$$\boxed{\mathcal{S} = \{\text{H-H, H-T, T-H, T-T}\}}$$

$$\begin{aligned} P_1^H &= \sum_{S \in \mathcal{S}} \Pr\{P_1^H \mid S\} \cdot \Pr\{S\} \\ &= P_1^H \cdot p^2 + 1 \cdot p(1-p) + P_1^H \cdot (1-p)p + 0 \cdot (1-p)^2 = p \end{aligned}$$

Second attempt: Cl-W vs. W-Cl (initial + 2 flips per turn)

Player 1: Cl-W | **Player 2: W-Cl**



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$$\boxed{P_1 = p^2 + (1-p)^2 = 2p^2 - 2p + 1}$$

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Second attempt: Cl-W vs. W-Cl (initial + 2 flips per turn)

Player 1: Cl-W

….......H H | T

Cl W

or

….......T T | H

Player 2: W-Cl

….......T H | H

W Cl

W Cl

Not fair!

P_1^H := probabil… first coin flip yields a H

$P_1 = P_1^H \cdot p + P_1^T \cdot (1 - \ldots$

$$\boxed{\mathcal{S} = \{\text{H-H, H-T, T-H, T-T}\}}$$

$$\boxed{P_1 = p^2 + (1-p)^2 = 2p^2 - 2p + 1}$$

$$P_1^H = \sum_{S \in \mathcal{S}} \Pr\{P_1^H \mid S\} \cdot \Pr\{S\}$$

$$= P_1^H \cdot p^2 + 1 \cdot p(1-p) + P_1^H \cdot (1-p)p + 0 \cdot (1-p)^2 = p$$

Third attempt: Cl-W vs. W-Cl (4 flips per turn)

Player 1: Cl-W

| H | H | T | H |

→ • Cl • W

or

| H | H | H | T |

→ • Cl • W

or

| T | T | T | H |

→ • Cl • W

or

| T | T | H | T |

→ • Cl • W

Player 2: W-Cl

| H | T | H | H |

→ • W • Cl

or

| T | H | H | H |

→ • W • Cl

or

| H | T | T | T |

→ • W • Cl

or

| T | H | T | T |

→ • W • Cl

Third attempt: Cl-W vs. W-Cl (4 flips per turn)

Player 1: Cl-W

H	H	T	H

• Cl • W

or

H	H	H	T

• Cl • W

or

T	T	T	H

• Cl • W

or

T	T	H	T

• Cl • W

Player 2: W-Cl

H	T	H	H

• W • Cl

or

T	H	H	H

• W • Cl

or

H	T	T	T

• W • Cl

or

T	H	T	T

• W • Cl

$$\text{expected \# flips until someone wins} = \frac{1}{-2p^4 + 4p^3 - 3p^2 + p}$$

Third attempt: Cl-W vs. W-Cl (4 flips per turn)

$$\text{expected \# flips until someone wins} = \frac{1}{-2p^4 + 4p^3 - 3p^2 + p}$$

Third attempt: Cl-W vs. W-Cl (4 flips per turn)

$$\text{expected \# flips until someone wins} = \frac{1}{-2p^4 + 4p^3 - 3p^2 + p}$$

$$\text{expected \# flips (regular von Neumann's)} = \frac{1}{p\,(1 - p)}$$

Third attempt: Cl-W vs. W-Cl (4 flips per turn)

expected # flips until someone wins $= \dfrac{1}{-2p^4 + 4p^3 - 3p^2 + p}$

$\div$

expected # flips (regular von Neumann's) $= \dfrac{1}{p\,(1-p)}$

$= \dfrac{1}{2p^2 - 2p + 1}$

for $0 < p < 1$, this means up to twice the number of flips!!

Our method: W*-Cl vs. W*-W (2 flips per turn + final)

Player 1: W*-Cl

..........H H

- W* Cl

or

..........T T

- W* Cl

Player 2: W*-W

..........H T

- W* W

or

..........T H

- W* W

W* := a whistle of even parity

Our method: W*-Cl vs. W*-W (2 flips per turn + final)

Player 1: W*-Cl

..........H H

• W* Cl

or

..........T T

• W* Cl

Player 2: W*-W

..........H T

• W* W

or

..........T H

• W* W

W* := a whistle of even parity

the three last flips

.... A | B C

W*

Our method: W*-Cl vs. W*-W (2 flips per turn + final)

Player 1: W*-Cl

.........H H

• W* Cl

or

.........T T

• W* Cl

Player 2: W*-W

.........H T

• W* W

or

.........T H

• W* W

W* := a whistle of even parity

the three last flips

.... A B C

W*

$$\Pr\{C \neq B \mid B \neq A\} = \Pr\{C = B \mid B \neq A\} = 0.5$$

Perfectly fair!

Our method: W*-Cl vs. W*-W (2 flips per turn + final)

Player 1: W*-Cl

.........H H

• W* Cl

or

.........T T

• W* Cl

Player 2: W*-W

.........H T

• W* W

or

.........T H

• W* W

W* := a whistle of even parity

W* occurs by the end of a "H-T" or a "T-H" turn

$\Pr\{W^*\} = 2\,p\,(1-p)$

expected # flips until someone wins $= 2 \times \dfrac{1}{2\,p\,(1-p)} + 1 = \boxed{\dfrac{1}{p\,(1-p)} + 1}$

Our method: W*-Cl vs. W*-W (2 flips per turn + final)

Player 1: W*-Cl

………H H

• W* Cl

or

………T T

• W* Cl

Player 2: W*-W

………H T

• W* W

or

………T H

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$\Pr\{W^*\} = 2\,p\,(1-p)$

expected # flips until someone wins $= 2 \times \dfrac{1}{2\,p\,(1-p)} + 1 =$  $+ 1$



Thank you!


Biased coins, blindfold players

Vinícius G. Pereira de Sá

based on the paper "Blind-friendly von Neumann's heads or tails",
to appear in The American Mathematical Monthly,
joint work with Celina M. H. de Figueiredo

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