Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges

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Abstract
Deciding whether an arbitrary graph admits a VLSI layout with unit-length edges is NP-complete [1], even when restricted to binary trees [7]. However, for certain graphs, the problem is polynomial or even trivial. A natural step, outstanding thus far, was to provide a broader classification of graphs that make for polynomial or NP-complete instances. We provide such a classification based on the set of vertex degrees in the input graphs, yielding a comprehensive dichotomy on the complexity of the problem, with and without the restriction to trees.

Keywords: graph, algorithm, VLSI, partial grid, NP-complete
A grid $G_{M \times N}$ has vertex set $V(G_{M \times N}) = \{(i, j) : 1 \leq i \leq M, 1 \leq j \leq N\}$, and edge set $E(G_{M \times N}) = \{(i, j)(k, l) : |i - k| + |j - l| = 1, (i, j), (k, l) \in V(G_{M \times N})\}$ (see Figure 1a). A grid embedding is a mapping from a graph’s vertices to a subset of the points of a grid, along with an incidence-preserving assignment of edges to non-crossing paths in the grid. Grid embeddings are widely studied in VLSI design and parallel architecture simulations [9,8].

A partial grid is any subgraph (not necessarily induced) of a grid, or, equivalently, a graph which admits an embedding with only unit-length edges.

Deciding whether a graph admits a unit-length embedding is NP-complete [1], even for binary trees [7]. Indeed, the so-called logic engine paradigm for proving the NP-hardness of problems in Graph Drawing is described in [4], where the seminal references [1,7] are discussed, along with further applications [5,6]. On the other hand, in the context of Graph Theory, the recognition of partial grids is often stated as an open problem [2,3].

This paper covers the Unit-Length VLSI problem (alternatively, Partial-Grid Recognition) when the input is restricted to $D$-graphs, for every possible set $D$ the degrees of the input vertices may belong to. Since the only connected graph with vertices of degree 0 is a singleton, and since graphs containing vertices of degree 5 or greater cannot possibly be embedded in a 2-dimensional, degree-4 grid, we are interested in the subsets of $\{1,2,3,4\}$.

Throughout the text, the term immersibility refers to a graph’s ability, or the lack thereof, to be embedded in a grid with only unit-length edges. All graphs considered in this paper are connected.
Fig. 2. Unit-length embedding for Bhatt and Cosmadakis’s extended skeleton $S_\varphi$ associated to the 3CNF formula $\varphi = (\overline{x_2} \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_4) \land (x_1 \lor x_3 \lor \overline{x_4})$.

1 Previous NP-completeness results

In [1], Bhatt and Cosmadakis proved that deciding the existence of unit-length embeddings for arbitrary trees is NP-complete. Their proof was based on the reduction of the well-known NP-complete problem NOT-ALL-EQUAL 3CNFSAT (not-all-equal conjunctive-normal-form satisfiability with 3 literals per clause) to the problem of deciding the existence of a unit-length embedding for a special tree they define, called the extended skeleton (see Figure 2). This problem is referred to as the Bhatt-Cosmadakis problem.

The seminal proof of Bhatt and Cosmadakis suffices to show that UNIT-LENGTH VLSI is NP-complete for $\{1,2,4\}$-trees, since the extended skeleton is itself a $\{1,2,4\}$-tree. It is also NP-complete for $\{1,2,3,4\}$-trees, since if the problem is NP-complete for $D$-trees, given a set $D$, then it is NP-complete for $D$-graphs (allowing cycles) and for $D'$-graphs, $D' \supset D$, as well.

The NP-completeness for $\{1,2,3\}$-trees was demonstrated by Gregori [7], who conceived an ingenious $\{1,2,3\}$-tree, called the $U$-tree, as a suitable replacement structure.

2 New NP-completeness results

We start with a new definition. Let $G$ be a graph. Say vertex $v \in G$ is adjacent to vertices $s$ and $t$. If, in all unit-length embeddings of $G$, edges $sv$ and $vt$ can only appear as two consecutive segments of the same grid line (or column), we say we have a pair of necessarily collinear edges. Analogously, if $sv$ and $vt$ can only be embedded with a $90^\circ$ angle between them, we say they are
necessarily orthogonal. If there is at least one unit-length embedding for $G$ in which $sv$ and $vt$ appear one way, and at least one unit-length embedding for $G$ in which they appear the other way, we say they constitute a pair of free-angle edges. In the graph of Figure 2, it is easy to see that edges $ax_{11}$ and $cx_{11}$ are necessarily collinear, whereas edges $ax_{11}$ and $bx_{11}$ are necessarily orthogonal, and all pairs of edges incident to a vertex painted black are free-angle.

Now we introduce a special $\{2,3\}$-graph called the double ladder. Figure 3a presents its only existing unit-length embedding. Vertices $x, y, z, w$ are regarded as interconnectors. Since the double ladder admits only one circular ordering of the interconnectors in all its feasible embeddings, the pairs of opposed interconnectors (namely $x, z$ and $w, y$) and of consecutive interconnectors (all other pairs) are fixed.

Let $G$ be a graph. We define the double-ladder substitution as the linear-time operation that obtains the graph $D(G)$ such that: (i) there is a bijection between each vertex $v$ in $G$ and a double ladder $d(v)$ in $D(G)$; and (ii) there is a bijection between each edge $uv$ in $G$ and an edge linking an interconnector of $d(u)$ to an interconnector of $d(v)$ in $D(G)$, in which case such interconnectors have become active. Figure 3b illustrates the result of a double-ladder substitution applied to the subgraph highlighted in Figure 2.

To preserve the immersibility of the original graph, it is mandatory that the choice of active interconnectors match the relative positions of all pairs of edges that are not free-angle in the original graph.
Lemma 2.1  Double-ladder substitution—with appropriately chosen interconnectors—preserves the immersibility of extended skeletons.

Proof. Let $S_\varphi$ be an extended skeleton. We show that $S_\varphi$ is a partial grid if and only if so is $D(S_\varphi)$. Suppose $D(S_\varphi)$ is a partial grid, and let $\Gamma'$ be a unit-length embedding of $D(S_\varphi)$. No matter how each double ladder is embedded, the distance between the centers of two adjacent double ladders is always 5. Since $S_\varphi$ is connected, the distance between the centers of any two double ladders in $\Gamma'$ is a multiple of 5 in both directions (vertical/horizontal). By substituting a single vertex $v$ (placed at its center) for each double ladder $d(v)$, and then depriving $\Gamma'$ of all lines and columns other than those containing the centers, we get a new grid that is 5 times smaller (on each dimension). Now, by adding to the new grid an edge $uv$ for every pair of adjacent double ladders $d(u), d(v)$, we obtain a unit-length embedding $\Gamma$ of $S_\varphi$.

For the converse, suppose $S_\varphi$ can be embedded in an $M \times N$ grid using unit-length edges, and let $\Gamma$ be such an embedding. Clearly, there will always be a unit-length embedding $\Gamma'$ for $D(S_\varphi)$ in a $5M \times 5N$ grid, where each vertex $v$ at coordinate $(i, j)$ in $\Gamma$ corresponds to a double ladder $d(v)$ spreading over a $5 \times 5$ square, in $\Gamma'$, whose center has coordinates $(5i, 5j)$. As for the connections between adjacent double ladders, we prove they can always be achieved by an appropriate choice of active interconnectors.

The point is, an extended skeleton is a rigid enough structure, so that all pairs of adjacent edges are either necessarily collinear or necessarily orthogonal. Thus, in order to make all the connections between adjacent double ladders in $\Gamma'$ possible, it suffices that, when connecting $d(v)$ to $d(s)$ and $d(t)$ (for $sv, vt \in S_\varphi$) in a double-ladder substitution on $S_\varphi$, we employ a pair of opposed interconnectors of $d(v)$ if $sv, vt$ are necessarily collinear, a pair of consecutive interconnectors if they are necessarily orthogonal, and an arbitrary pair of interconnectors of $d(v)$ if they are free-angle.

Theorem 2.2  Unit-Length VLSI is NP-complete for $\{2,3\}$-graphs.

Proof. Clearly, Unit-Length VLSI belongs to NP, regardless of the input. Now, since Bhatt-Cosmadakis is NP-complete (see Section 1) and, by Lemma 2.1, any extended skeleton can be polynomially transformed—via double-ladder substitution—into a $\{2,3\}$-graph with the same immersibility, Unit-Length VLSI is NP-complete when restricted to $\{2,3\}$-graphs as well. As $\{2,3,4\} \supset \{2,3\}$, the NP-completeness for $\{2,3,4\}$-graphs follows.

We also prove the problem is NP-complete for $\{1,3\}$- and $\{2,4\}$-graphs. The proof, whose details were left for the full version of this paper due to space
NP-completeness results for \{1, 3, 4\}-trees and \{2, 3, 4\}-graphs follow directly from the superset property.

3 Polynomially decidable cases

For \{1\}-, \{2\}- and \{1, 2\}-graphs the problem is trivial. A path on \(n\) vertices can always be laid out on a straight line of a \(1 \times n\) grid, and any even cycle
on $2k$ vertices can be embedded on a $2 \times k$ grid. Odd cycles are not bipartite and therefore cannot be partial grids.

The problem is also trivial for $\{3\}$-, $\{4\}$- and $\{3,4\}$-graphs, for these graphs can never be partial grids. Suppose there is a unit-length embedding $\Gamma$ for a graph with no vertices of degree 1 or 2. Let $v$ be the topmost vertex in the leftmost column of $\Gamma$. Since all other vertices are placed below or to the right of $v$, $v$ can have at most 2 neighbors, a contradiction.

An interesting polynomial case is that of $\{1,4\}$-graphs, which completes our dichotomy.

**Theorem 3.1** A $\{1,4\}$-graph is a partial grid if and only if its degree-4 vertices induce a grid graph.

**Proof.** Let $G$ be a $\{1,4\}$-graph, and $G'$ the subgraph of $G$ induced by all its vertices of degree 4. If $G'$ is a grid, then there is always a unit-length embedding for $G$, in which the degree-4 vertices occupy all points of an $M \times N$ rectangle, surrounded by the $2(M+N)$ degree-1 vertices, which are necessarily adjacent to the vertices in the boundaries of such rectangle. For the converse, whose details are omitted here, the idea is that, if $G'$ is a connected partial grid that is not a grid, any embedding of $G'$ must present a unit-area square $\sigma$ containing at least 2 but no more than 3 edges of $G'$. This, along with the fact that the vertices of $G'$ have degree 4 in $G$, leads to a contradiction. \qed

### 4 Conclusion and open problems

Please refer to Figure 6 for the summarized complexity dichotomy. Existing results are duly referenced.

A question of theoretical interest concerns the existence of replacement $D$-graphs that always preserve immersibility. The gadgets introduced herein, albeit sufficient for the intended proofs, do not guarantee that the immersibility of the original graph is preserved when the relative positions of its edges are not known beforehand. Another question worth considering is how the complexities get affected by allowing edges with length up to $k \geq 1$.

### References


[2] Brandstädt, A., Le, V. B., Szymczak, T., Siegemund, F., de Ridder,
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Fig. 6. Complexity dichotomy ("NP-C": NP-complete; "P": polynomial; "—": the corresponding input does not exist).


