Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges

Vinícius G. P. de Sá   Celina M. H. de Figueiredo

Universidade Federal do Rio de Janeiro, Brasil
vigusmao@dcc.ufrj.br celina@cos.ufrj.br

Guilherme D. da Fonseca

Universidade Federal do Estado do Rio de Janeiro, Brasil
fonseca@uniriotec.br

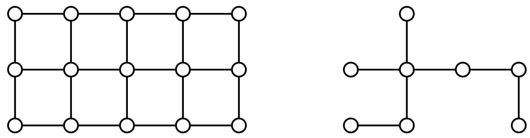
Raphael Machado

Instituto Nacional de Metrologia, Normalização e Qualidade Industrial
Universidade Federal do Rio de Janeiro, Brasil
raphael@cos.ufrj.br

Abstract

Deciding whether an arbitrary graph admits a VLSI layout with unit-length edges is NP-complete [1], even when restricted to binary trees [7]. However, for certain graphs, the problem is polynomial or even trivial. A natural step, outstanding thus far, was to provide a broader classification of graphs that make for polynomial or NP-complete instances. We provide such a classification based on the set of vertex degrees in the input graphs, yielding a comprehensive dichotomy on the complexity of the problem, with and without the restriction to trees.

Keywords: graph, algorithm, VLSI, partial grid, NP-complete



(a) (b)

Fig. 1. (a) The grid $G_{3,5}$. (b) Unit-length embedding for a $\{1, 2, 4\}$-tree.

A *grid* $G_{M\times N}$ has vertex set $V(G_{M\times N}) = \{(i, j) : 1 \leq i \leq M, 1 \leq j \leq N\}$, and edge set $E(G_{M\times N}) = \{(i, j)(k, l) : |i - k| + |j - l| = 1, (i, j), (k, l) \in V(G_{M\times N})\}$ (see Figure 1a). A *grid embedding* is a mapping from a graph's vertices to a subset of the points of a grid, along with an incidence-preserving assignment of edges to non-crossing paths in the grid. Grid embeddings are widely studied in VLSI design and parallel architecture simulations [9,8]. A *partial grid* is any subgraph (not necessarily induced) of a grid, or, equivalently, a graph which admits an embedding with only unit-length edges.

Deciding whether a graph admits a unit-length embedding is NP-complete [1], even for binary trees [7]. Indeed, the so-called *logic engine paradigm* for proving the NP-hardness of problems in Graph Drawing is described in [4], where the seminal references [1,7] are discussed, along with further applications [5,6]. On the other hand, in the context of Graph Theory, the recognition of partial grids is often stated as an open problem [2,3].

We consider the complexity dichotomy into polynomial and NP-complete for degree-constrained VLSI layouts with unit-length edges. Let D be a set of non-negative integers. We say a graph is a *D-graph* if the degrees of all its vertices are elements of D, e.g. a path is a {1,2}-graph, a cycle is a {2}-graph, a complete graph on n vertices is a $\{n - 1\}$-graph etc. Notice that a D-graph G is also a D'-graph, for $D' \supset D$, since it is not required that all elements of D actually appear as the degree of some vertex in G (see Figure 1b).

This paper covers the Unit-Length VLSI problem (alternatively, Partial-Grid Recognition) when the input is restricted to D-graphs, for every possible set D the degrees of the input vertices may belong to. Since the only connected graph with vertices of degree 0 is a singleton, and since graphs containing vertices of degree 5 or greater cannot possibly be embedded in a 2-dimensional, degree-4 grid, we are interested in the subsets of {1,2,3,4}.

Throughout the text, the term *immersibility* refers to a graph's ability, or the lack thereof, to be embedded in a grid with only unit-length edges. All graphs considered in this paper are connected.

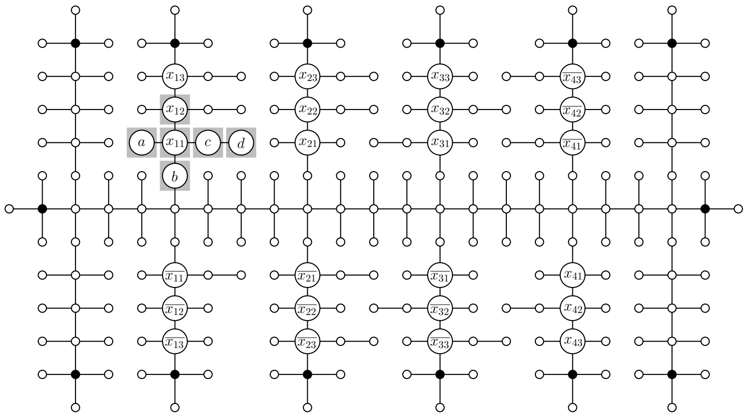


Fig. 2. Unit-length embedding for Bhatt and Cosmadakis's extended skeleton S_φ associated to the 3CNF formula $\varphi = (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_4})$.

1 Previous NP-completeness results

In [1], Bhatt and Cosmadakis proved that deciding the existence of unit-length embeddings for arbitrary trees is NP-complete. Their proof was based on the reduction of the well-known NP-complete problem Not-All-Equal 3CNFSAT (not-all-equal conjunctive-normal-form satisfiability with 3 literals per clause) to the problem of deciding the existence of a unit-length embedding for a special tree they define, called the *extended skeleton* (see Figure 2). This problem is referred to as the Bhatt-Cosmadakis problem.

The seminal proof of Bhatt and Cosmadakis suffices to show that Unit-Length VLSI is NP-complete for {1,2,4}-trees, since the extended skeleton is itself a {1,2,4}-tree. It is also NP-complete for {1,2,3,4}-trees, since if the problem is NP-complete for D-trees, given a set D, then it is NP-complete for D-graphs (allowing cycles) and for D'-graphs, $D' \supset D$, as well.

The NP-completeness for {1,2,3}-trees was demonstrated by Gregori [7], who conceived an ingenious {1,2,3}-tree, called the *U-tree*, as a suitable replacement structure.

2 New NP-completeness results

We start with a new definition. Let G be a graph. Say vertex $v \in G$ is adjacent to vertices s and t. If, in all unit-length embeddings of G, edges sv and vt can only appear as two consecutive segments of the same grid line (or column), we say we have a pair of *necessarily collinear* edges. Analogously, if sv and vt can only be embedded with a 90° angle between them, we say they are

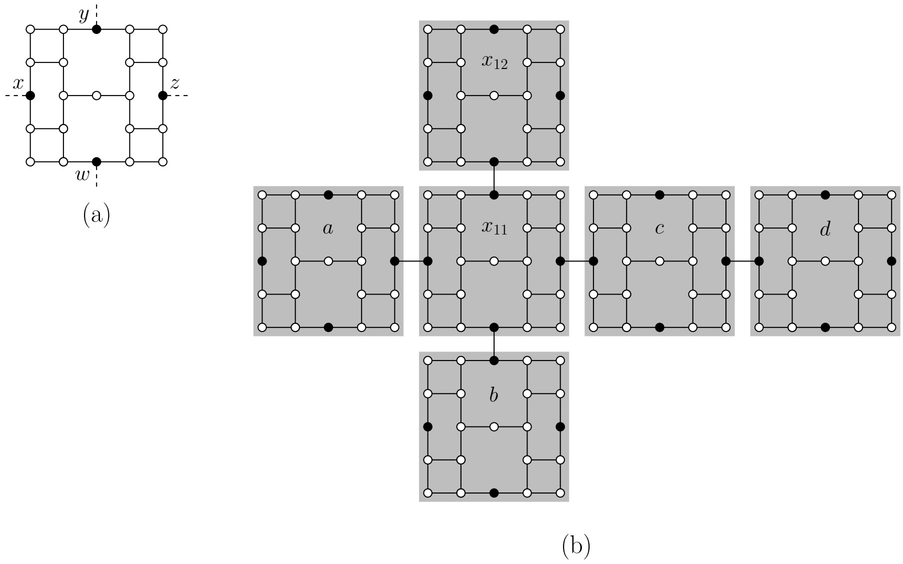


Fig. 3. (a) The {2,3} gadget (double ladder). (b) Double-ladder substitution.

necessarily orthogonal. If there is at least one unit-length embedding for G in which sv and vt appear one way, and at least one unit-length embedding for G in which they appear the other way, we say they constitute a pair of *free-angle* edges. In the graph of Figure 2, it is easy to see that edges ax_{11} and cx_{11} are necessarily collinear, whereas edges ax_{11} and bx_{11} are necessarily orthogonal, and all pairs of edges incident to a vertex painted black are free-angle.

Now we introduce a special {2,3}-graph called the *double ladder*. Figure 3a presents its only existing unit-length embedding. Vertices x, y, z, w are regarded as *interconnectors*. Since the double ladder admits only one circular ordering of the interconnectors in all its feasible embeddings, the pairs of *opposed* interconnectors (namely x, z and w, y) and of *consecutive* interconnectors (all other pairs) are fixed.

Let G be a graph. We define the *double-ladder substitution* as the linear-time operation that obtains the graph $D(G)$ such that: (i) there is a bijection between each vertex v in G and a double ladder $d(v)$ in $D(G)$; and (ii) there is a bijection between each edge uv in G and an edge linking an interconnector of $d(u)$ to an interconnector of $d(v)$ in $D(G)$, in which case such interconnectors have become *active*. Figure 3b illustrates the result of a double-ladder substitution applied to the subgraph highlighted in Figure 2.

To preserve the immersibility of the original graph, it is mandatory that the choice of active interconnectors match the relative positions of all pairs of edges that are not free-angle in the original graph.

Lemma 2.1 *Double-ladder substitution—with appropriately chosen interconnectors—preserves the immersibility of extended skeletons.*

Proof. Let S_φ be an extended skeleton. We show that S_φ is a partial grid if and only if so is $D(S_\varphi)$. Suppose $D(S_\varphi)$ is a partial grid, and let Γ' be a unit-length embedding of $D(S_\varphi)$. No matter how each double ladder is embedded, the distance between the centers of two adjacent double ladders is always 5. Since S_φ is connected, the distance between the centers of *any* two double ladders in Γ' is a multiple of 5 in both directions (vertical/horizontal). By substituting a single vertex v (placed at its center) for each double ladder $d(v)$, and then depriving Γ' of all lines and columns other than those containing the centers, we get a new grid that is 5 times smaller (on each dimension). Now, by adding to the new grid an edge uv for every pair of adjacent double ladders $d(u), d(v)$, we obtain a unit-length embedding Γ of S_φ.

For the converse, suppose S_φ can be embedded in an $M \times N$ grid using unit-length edges, and let Γ be such an embedding. Clearly, there will always be a unit-length embedding Γ' for $D(S_\varphi)$ in a $5M \times 5N$ grid, where each vertex v at coordinate (i, j) in Γ corresponds to a double ladder $d(v)$ spreading over a 5×5 square, in Γ', whose center has coordinates $(5i, 5j)$. As for the connections between adjacent double ladders, we prove they can always be achieved by an appropriate choice of active interconnectors.

The point is, an extended skeleton is a rigid enough structure, so that all pairs of adjacent edges are either necessarily collinear or necessarily orthogonal. Thus, in order to make all the connections between adjacent double ladders in Γ' possible, it suffices that, when connecting $d(v)$ to $d(s)$ and $d(t)$ (for $sv, vt \in S_\varphi$) in a double-ladder substitution on S_φ, we employ a pair of opposed interconnectors of $d(v)$ if sv, vt are necessarily collinear, a pair of consecutive interconnectors if they are necessarily orthogonal, and an arbitrary pair of interconnectors of $d(v)$ if they are free-angle. □

Theorem 2.2 UNIT-LENGTH VLSI *is NP-complete for* $\{2, 3\}$*-graphs.*

Proof. Clearly, UNIT-LENGTH VLSI belongs to NP, regardless of the input. Now, since BHATT-COSMADAKIS is NP-complete (see Section 1) and, by Lemma 2.1, any extended skeleton can be polynomially transformed—via double-ladder substitution—into a {2,3}-graph with the same immersibility, UNIT-LENGTH VLSI is NP-complete when restricted to {2,3}-graphs as well. As $\{2, 3, 4\} \supset \{2, 3\}$, the NP-completeness for {2,3,4}-graphs follows. □

We also prove the problem is NP-complete for $\{1, 3\}$- and $\{2, 4\}$-graphs. The proof, whose details were left for the full version of this paper due to space

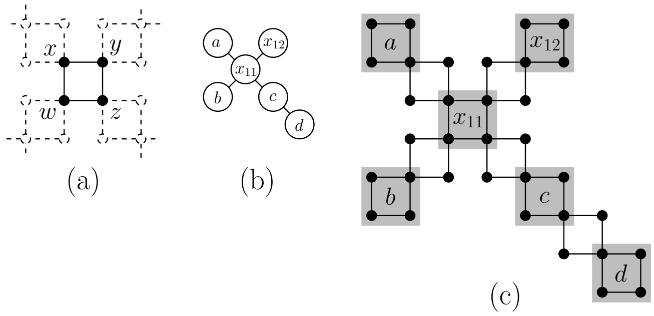


Fig. 4. (a) The {2,4} gadget. (b) Rotation of 45°. (c) Square substitution.

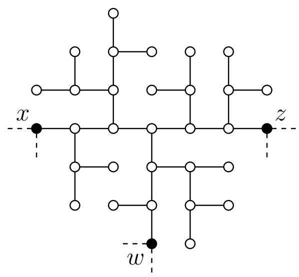


Fig. 5. The {1,3} gadget.

constraints, is based on analogous immersibility-preserving transformations. For $\{2, 4\}$-graphs, the gadget is a simple C_4 (a square), with the peculiarity that it replaces not only the vertices but also the edges of the extended skeleton being transformed, in what we call a *square substitution*. Existing layouts of square-substituted graphs look as though the corresponding layouts of the original graphs had been rotated 45°, as illustrated in Figure 4a-c. The gadget for $\{1, 3\}$-graphs is actually a special $\{1, 3\}$-tree (see Figure 5) which must be used in combination with Gregori's U-tree, since it has only three interconnectors. The original extended skeleton is therefore initially transformed into a $\{1, 2, 3\}$-tree by having its vertices replaced with U-trees, and only then transformed into a $\{1, 3\}$-tree by using an immersibility-preserving substitution which employs the new gadget.

NP-completeness results for $\{1, 3, 4\}$-trees and $\{2, 3, 4\}$-graphs follow directly from the the superset property.

3 Polynomially decidable cases

For $\{1\}$-, $\{2\}$- and $\{1, 2\}$-graphs the problem is trivial. A path on n vertices can always be laid out on a straight line of a $1 \times n$ grid, and any even cycle on $2k$ vertices can be embedded on a $2 \times k$ grid. Odd cycles are not bipartite and therefore cannot be partial grids.

The problem is also trivial for {3}-, {4}- and {3,4}-graphs, for these graphs can never be partial grids. Suppose there is a unit-length embedding Γ for a graph with no vertices of degree 1 or 2. Let v be the topmost vertex in the leftmost column of Γ. Since all other vertices are placed below or to the right of v, v can have at most 2 neighbors, a contradiction.

An interesting polynomial case is that of $\{1, 4\}$-graphs, which completes our dichotomy.

Theorem 3.1 *A {1,4}-graph is a partial grid if and only if its degree-4 vertices induce a grid graph.*

Proof. Let G be a {1,4}-graph, and G' the subgraph of G induced by all its vertices of degree 4. If G' is a grid, then there is always a unit-length embedding for G, in which the degree-4 vertices occupy all points of an $M \times N$ rectangle, surrounded by the $2(M+N)$ degree-1 vertices, which are necessarily adjacent to the vertices in the boundaries of such rectangle. For the converse, whose details are omitted here, the idea is that, if G' is a connected partial grid that is *not* a grid, any embedding of G' must present a unit-area square σ containing at least 2 but no more than 3 edges of G'. This, along with the fact that the vertices of G' have degree 4 in G, leads to a contradiction. □

4 Conclusion and open problems

Please refer to Figure 6 for the summarized complexity dichotomy. Existing results are duly referenced.

A question of theoretical interest concerns the existence of replacement D-graphs that always preserve immersibility. The gadgets introduced herein, albeit sufficient for the intended proofs, do not guarantee that the immersibility of the original graph is preserved when the relative positions of its edges are not known beforehand. Another question worth considering is how the complexities get affected by allowing edges with length up to $k > 1$.

References

[1] Bhatt, S. N., and Cosmadakis, S. S., *The complexity of minimizing wire lengths in VLSI layouts*, Inform. Process. Lett. **25** (1987), 263–267.

[2] Brandstädt, A., Le, V. B., Szymczak, T., Siegemund, F., de Ridder,

D	D-graphs	D-trees
{1}	P	P
{2}	P	—
{3}	P	—
{4}	P	—
{1,2}	P	P
{1,3}	NP-C	NP-C
{1,4}	P	P
{2,3}	NP-C	—

D	D-graphs	D-trees
{2,4}	NP-C	—
{3,4}	P	—
{1,2,3}	NP-C [7]	NP-C [7]
{1,2,4}	NP-C [1]	NP-C [1]
{1,3,4}	NP-C	NP-C
{2,3,4}	NP-C	—
{1,2,3,4}	NP-C [1]	NP-C [1]

Fig. 6. Complexity dichotomy ("NP-C": NP-complete; "P": polynomial; "—": the corresponding input does not exist).

H. N., Knorr, S., Rzehak, M., Mowitz, M., Ryabova, N., and Nagel, U., *Information system on graph class inclusions*, WWW document at `http://wwwteo.informatik.uni-rostock.de/isgci/about.html`, 2002.

[3] Campos, C. N., and Mello, C. P., *The total chromatic number of some bipartite graphs*, Proc. 7th International Colloquium on Graph Theory, 557–561, Electron. Notes Discrete Math. **22**, 2005. Full paper in Ars Combin. **88** (2008), 335–347.

[4] Di Battista, G., Eades, P., Tamassia, R., and Tollis, I.G., "Graph Drawing – Algorithms for the Visualization of Graphs" (Prentice Hall, 1999).

[5] P. Eades, S. Whitesides, The realization problem for Euclidean minimum spanning trees is NP-hard. Algorithmica 16 (1996), 60–82.

[6] P. Eades, S. Whitesides, The logic engine and the realization problem for nearest neighbor graphs. Theoret. Comput. Sci. 169 (1996), 23–37.

[7] Gregori, A., *Unit-length embedding of binary trees on a square grid*, Inform. Process. Lett. **31** (1989), 167–173.

[8] Lin, Y. B., Miller, Z., Perkel, M., Pritikin, D., and Sudborough, I. H., *Expansion of layouts of complete binary trees into grids*, Discrete Appl. Math. **131** (2003), 611–642

[9] Ullman, J. D., Computational Aspects of VLSI (Addison-Wesley, 1984).