Geodetic number versus hull number in P_3 convexity*

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Abstract

We study the graphs G for which the hull number $h(G)$ and the geodetic number $g(G)$ with respect to P_3-convexity coincide. These two parameters correspond to the minimum cardinality of a set U of vertices of G such that the simple expansion process which iteratively adds to U all vertices outside of U having two neighbors in U produces the whole vertex set of G either eventually or after one iteration, respectively. We establish numerous structural properties of the graphs G with $h(G) = g(G)$, allowing for the constructive characterization as well as the efficient recognition of all such graphs that are triangle-free. Furthermore, we characterize—in terms of forbidden induced subgraphs—the graphs G that satisfy $h(G') = g(G')$ for every induced subgraph G' of G.

Keywords. Hull number; geodetic number; P_3-convexity; irreversible 2-threshold processes

*This is a manuscript by the authors. The final article was published by SIAM with DOI number 10.1137/110859014.

1 Introduction

As one of the most elementary models of spreading a property within a network—like sharing an idea or disseminating a virus—one can consider a graph G, a set U of vertices of G that initially possess the property, and an iterative process whereby new vertices u enter U whenever sufficiently many neighbors of u are already in U. The simplest choice for "sufficiently many" that results in interesting effects is two. This choice leads to the *irreversible* 2*-threshold processes* considered by Dreyer and Roberts [8]. Similar models were studied in various contexts such as statistical physics, distributed computing, social networks, and marketing, under different names such as influence dynamics, bootstrap percolation, local majority processes, irreversible dynamic monopolies, catastrophic fault patterns, and many others [1, 2, 4, 8, 11, 13–15].

From the point of view of discrete convexity, the above spreading process corresponds to the formation of the convex hull of the set U of vertices of G with respect to the so-called P_3*-convexity* in G. A set C of vertices of G is P_3*-convex* if no vertex of G outside of C has two neighbors in C, and the P_3*-convex hull* of a set U of vertices of G is the smallest P_3-convex set containing U. A P_3*-hull set* of G is a set of vertices whose P_3-convex hull equals the whole vertex set of G, and the minimum cardinality of a P_3-hull set of G is the P_3*-hull number* $h(G)$ of G. Closely related to the notion of hull sets and the hull number of a graph are geodetic sets and the geodetic number. A P_3*-geodetic set* of a graph G is a set U of vertices of G such that every vertex of G outside of U has two neighbors in U. The minimum cardinality of a P_3-geodetic set of G is the P_3*-geodetic number* $g(G)$ of G.

Various types of graph convexities have been considered in the literature, and the definitions of hull sets and geodetic sets change accordingly. For the special case of the P_3-convexity, the P_3-geodetic number coincides with the well-studied 2-domination number [10]. The P_3-convexity was first considered for tournaments [9, 12, 16], and by now many aspects of the P_3-convexity in graphs have been studied, such as partition problems [5] and versions of Carathéodory's theorem [3] and Radon's theorem [7].

In view of the iterative spreading process considered above, a hull set *eventually* distributes the property throughout the entire network, whereas a geodetic set spreads the property within the entire network *in exactly one iteration*. Hence the hull number $h(G)$ of some graph G is the minimum size of an initial set needed to spread the property throughout G without a limit on the number of iterations, while the geodetic number $g(G)$ is the minimum size of an initial set needed to spread the property throughout G in one single iteration. Intuitively, there should be a tradeoff between the speed of the spreading process and the size of the initial set—the faster one wants the process to terminate, the larger the initial set should have to be. This intuition is reflected by the inequality

$$h(G) \leq g(G), \tag{1}$$

which holds for every graph G. A formal proof of (1) follows immediately from the trivial observation that every geodetic set is a hull set.

While a tradeoff is intuitively plausible, it does not necessarily occur for all network topologies G. On one hand, spreading the property immediately instead of eventually requires at most $g(G) - h(G)$ more vertices in the initial set. Therefore, if $h(G)$ and $g(G)$ are close together, then the speedup of the spreading process comes at a small additional price as measured in the size of the initial set. On the other hand, one may be interested in reducing as much as possible the number of "initially infected" elements, yet allowing for the dissemination to eventually reach the entire network, whereupon $g(G) - h(G)$ bounds the maximum possible such reduction. In order to understand this time versus size of the initial set tradeoff, it is of interest to study graphs G for which $h(G)$ and $g(G)$ are close together. Since both parameters are computationally hard in general and efficient algorithms are only known for quite restricted graph classes [6,10], it is most likely algorithmically hard to decide for a given graph G whether $h(G)$ and $g(G)$ are close together.

In the present paper we study the extreme case of graphs G where $h(G)$ and $g(G)$ coincide, that is, (1) holds with equality, and no speedup—or reduction in the size of the initial set—is possible. After summarizing useful notation and terminology, we collect numerous structural properties of such graphs in Section 2. Based on these properties, we construct a large subclass of those graphs in Section 3, comprising all such graphs that are triangle-free. In Section 4 we derive an efficient algorithm for the recognition of the triangle-free graphs that satisfy (1) with equality. In Section 5 we give a complete characterization of the class of all graphs G for which (1) holds with equality for every induced subgraph of G. Finally, we conclude with some open problems in Section 6.

1.1 Notation and terminology

We consider finite and simple graphs and digraphs, and use standard terminology. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. For a vertex u of a graph G, the neighborhood of u in G is denoted $N_G(u)$ and the degree of u in G is denoted $d_G(u)$. A set C of vertices of G is *P_3-convex* exactly if no vertex of G outside C has two neighbors in C. The *P_3-convexity of G* is the collection $\mathcal{C}(G)$ of all P_3-convex sets. Since we only consider P_3-convexity, we will omit the prefix "P_3-" from now on.

For a set U of vertices of G, let the *interval $I_G(U)$ of U in G* be the set $U \cup \{u \in V(G) \setminus U \mid |N_G(u) \cap U| \geq 2\}$, and let $H_G(U)$ denote the *convex hull of U in G*, that is, $H_G(U)$ is the unique smallest set in $\mathcal{C}(G)$ containing U. Within this notation, U is a geodetic set of G if $I_G(U) = V(G)$, and U is a hull set of G if $H_G(U) = V(G)$. The inequality (1) follows from the immediate observation that $I_G(U) \subseteq H_G(U)$ for every set U.

If U is a hull set of G, then iteratively adding vertices to U that have two neighbors in U results in $V(G)$. This defines a linear order $u_1, \ldots, u_n$ of the vertices of G such that $U = \{u_1, \ldots, u_{|U|}\}$ and for every i with $|U| + 1 \leq i \leq n$, the vertex u_i has two neighbors in $\{u_1, \ldots, u_{i-1}\}$. This implies that U is a hull set of G if and only if there is an acyclic orientation D of a spanning subgraph of G such that the in-degree $d_D^-(u)$ is 0 for every vertex u in U and 2 for every vertex u in $V(G) \setminus U$. We call such a D a *hull proof for U in G*.

Throughout the paper we will use the term component to denote a connected component. Since the hull number and the geodetic number are both additive with respect to the components of G, we consider the set of graphs

$$\mathcal{H} = \{G \mid G \text{ is a connected graph with } h(G) = g(G)\}.$$

2 Structural properties of graphs in $\mathcal{H}$

We collect some structural properties of the graphs in $\mathcal{H}$. Throughout this section, let G be a fixed graph in $\mathcal{H}$. Let W be a geodetic set of G of minimum order and let $B = V(G) \setminus W$. By definition, every vertex in B has at least two neighbors in W. Therefore, G has a spanning bipartite subgraph G_0 with bipartition $V(G_0) = W \cup B$ such that every vertex in B has degree exactly 2 in G_0. Let E_1 denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in the same component of G_0 and let E_2 denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in distinct components of G_0. Note that, by construction, W is a geodetic set of G_0. Since $|W| = g(G) = h(G) \leq h(G_0) \leq g(G_0) \leq |W|$, we obtain $h(G_0) = g(G_0) = |W|$, that is, G_0 has no geodetic set and no hull set of order less than $|W|$. Thus, if C is a component of G_0, then $W \cap V(C)$ is a minimum geodetic set of C as well as a minimum hull set of C.

Lemma 1 *Let C be a component of G_0.*

(i) No two vertices in C are incident with edges in E_2.

(ii) If some vertex u in C is incident with at least two edges in E_2, then u belongs to B and u is a cut vertex of C.

Proof (i) We consider different cases. If two vertices w and w' in $V(C) \cap W$ are incident with edges in E_2, then let $P : w_1b_1 \dots w_{l-1}b_{l-1}w_l$ be a shortest path in C between $w = w_1$ and $w' = w_l$. The set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

If a vertex w in $V(C) \cap W$ and a vertex b in $V(C) \cap B$ are incident with edges in E_2, then let $P : w_1b_1 \dots w_lb_l$ be a shortest path in C between $w = w_1$ and $b = b_l$. Note that b has a neighbor in G_0 that does not belong to P. Therefore, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

Finally, if two vertices b and b' in $V(C) \cap B$ are incident with edges in E_2, then let $P : b_1w_1 \dots b_{l-1}w_{l-1}b_l$ be a shortest path in C between $b = b_1$ and $b' = b_l$. Note that b and b' both have neighbors in G_0 that do not belong to P. Therefore, the set $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

(ii) If a vertex w in $V(C) \cap W$ is incident with at least two edges in E_2, then $W \setminus \{w\}$ is a hull set of G, which is a contradiction.

If a vertex b in $V(C) \cap B$ that is not a cut vertex of C is incident with at least two edges in E_2, then let $P : w_1b_1 \dots w_{l-1}b_{l-1}w_l$ be a path in C avoiding b between the two neighbors w_1 and w_l of b in G_0. The set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 2 *If G_0 is not connected, no two vertices in W that belong to the same component of G_0 are adjacent.*

Proof: For contradiction, we assume that ww' is an edge of G where w and w' are vertices in W that belong to the same component C of G_0. Since G is connected, there is an edge uv in E_2 with $u \in V(C)$ and $v \in V(G) \setminus V(C)$.

First, we assume that u belongs to W. Let $P : w_1b_1 \dots w_{l-1}b_{l-1}w_l$ be a shortest path in C between $u = w_1$ and a vertex w_l in $\{w, w'\}$. Note that $l = 1$ is possible. The set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

Next, we assume that u belongs to B. Let $P : b_1w_1 \dots b_lw_l$ be a shortest path in C between $u = b_1$ and a vertex w_l in $\{w, w'\}$. Note that $l = 1$ is possible. Furthermore, note that b_1 has a neighbor in G_0 that does not belong to P. The set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_2, \dots, b_l\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 3 *If G_0 is not connected and C is a component of G_0, then there are no two vertices w in $V(C) \cap W$ and b in $V(C) \cap B$ such that $wb \in E_1$.*

Proof: For contradiction, we assume that there is an edge $wb \in E_1$ with w in W and b in B. Since G is connected, there is an edge uv in E_2 with $u \in V(C)$ and $v \in V(G) \setminus V(C)$.

First, we assume that $u \in W$. Let P be a shortest path in C between u and a vertex u' in $\{w, b\}$. If $u' = w$, then let $P : w_1 b_1 \dots b_{l-1} w_l$ where $u = w_1$ and $w = w_l$. Note that $l = 1$ is possible. In this case the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. If $u' = b$, then let $P : w_1 b_1 \dots b_{l-1} w_l b_l$ where $u = w_1$ and $b = b_l$. Note that $l = 1$ is possible. Furthermore, note that b has a neighbor in G_0 that does not belong to P. In this case, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

Next, we assume that $u = b$. Let $P : b_1 w_1 \dots b_l w_l$ be a shortest path in C between $b = b_1$ and $w = w_l$. Note that the edge bw does not belong to C, hence $l \geq 2$. Furthermore, note that b has a neighbor in G_0 that does not belong to P. In this case, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_2, \dots, b_l\}$ is a hull set of G, which is a contradiction.

Finally, we assume that $u \in B \setminus \{b\}$. Let P be a shortest path in C between u and a vertex u' in $\{w, b\}$. If $u' = w$, then let $P : b_1 w_1 \dots b_l w_l$, where $u = b_1$ and $w = w_l$. Note that $l = 1$ is possible. Furthermore, note that w is the unique neighbor of b in P, and that u has a neighbor in G_0 that does not belong to P. In this case, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_2, \dots, b_l\}$ is a hull set of G, which is a contradiction. If $u' = b$, then let $P : b_1 w_1 \dots w_{l-1} b_l$, where $u = b_1$ and $b = b_l$. In this case, the set $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 4 *Let G_0 be disconnected and let b and b' be two vertices in B that belong to the same component C of G_0 satisfying $bb' \in E_1$.*

(i) Neither b nor b' is incident with an edge in E_2.

(ii) If some vertex w in $V(C) \cap W$ is incident with an edge in E_2 and $P : w_1 b_1 \dots w_l b_l$ is a path in C between $w = w_1$ and a vertex b_l in $\{b, b'\}$, then w_l is adjacent to both b and b', and C contains no path between b and b' that does not contain w_l.

(iii) If some vertex b'' in $(V(C) \cap B) \setminus \{b, b'\}$ is incident with an edge in E_2 and $P : b_1 w_1 \dots w_{l-1} b_l$ is a path in C between $b'' = b_1$ and a vertex b_l in $\{b, b'\}$, then w_{l-1} is adjacent to both b and b' and C contains no path between b and b' that does not contain w_{l-1}.

Proof: (i) For contradiction, we assume that b' is incident with an edge in E_2. If b and b' have a common neighbor w in C, then, since $bb' \in E_1$, the set $W \setminus \{w\}$ is a hull set of G, which is a contradiction. Hence we may assume that b and b' do not have a common neighbor in C. Let w_1 be a neighbor of b in G_0 and w_l a neighbor of b' in G_0 chosen in such a way that the path $P : w_1 b_1 \dots w_{l-1} b_{l-1} w_l$ in C is shortest possible. Note that b and b' both have neighbors in G_0 that do not belong to P. Therefore and since $bb' \in E_1$, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

(ii) Let w and $P : w_1 b_1 \dots w_l b_l$ be as specified.

First, we assume, for contradiction, that w_l is not adjacent to both b and b'. In this case, $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. Hence w_l is adjacent to both b and b' and we may assume that $b_l = b$.

Next, we assume, for contradiction, that C contains a path $Q : b'_1w'_1 \dots w'_{k-1}b'_k$ between $b = b'_1$ and $b' = b'_k$ that does not contain w_l. If $k = 2$, then w'_1 is a common neighbor of b and b' in W distinct from w_l, and $(W \setminus \{w_1, \dots, w_l, w'_1\}) \cup \{b_1, \dots, b_l\}$ is a hull set of G, which is a contradiction. Hence $k \geq 3$, which implies that w_l is the only common neighbor of b and b' in C. If the two paths P and Q intersect in a vertex in W, say $w_i = w'_j$, then $w_1b_1 \dots b_{i-1}w_ib'_{j+1}w'_{j+1} \dots w'_{k-1}b'_k$ is a path in C between w and a vertex in $\{b, b'\}$ such that w'_{k-1} is not adjacent to both b and b', which leads to a contradiction as above. Hence we may assume that P and Q do not intersect in a vertex in W. Now $\big(W \setminus (\{w_1, \dots, w_l\} \cup \{w'_1, \dots, w'_{k-1}\})\big) \cup \{b_1, \dots, b_l\} \cup \{b'_2, \dots, b'_{k-1}\}$ is a hull set of G, which is a contradiction.

(iii) Let b'' and $P : b_1w_1 \dots w_{l-1}b_l$ be as specified.

If b_1 is adjacent to some vertex w_i of P that is distinct from w_1, then we can shorten P to $b_1w_ib_{i+1} \dots w_{l-1}b_l$ without changing the neighbor w_{l-1} of b_l on P. Hence we may assume that b'' has a neighbor in G_0 that does not belong to P.

First, we assume, for contradiction, that w_{l-1} is not adjacent to both b and b'. In this case, the set $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. Hence w_{l-1} is adjacent to both b and b' and we may assume that $b_l = b$.

Next, we assume, for contradiction, that C contains a path $Q : b'_1w'_1 \dots w'_{k-1}b'_k$ between $b = b'_1$ and $b' = b'_k$ that does not contain w_{l-1}. If $k = 2$, then w'_1 is a common neighbor of b and b' in W distinct from w_l and $(W \setminus \{w_1, \dots, w_{l-1}, w'_1\}) \cup \{b_2, \dots, b_l\}$ is a hull set of G, which is a contradiction. Hence $k \geq 3$, which implies that w_{l-1} is the only common neighbor of b and b' in C. If the two paths P and Q intersect in a vertex in W, then we obtain a similar contradiction as in (ii). Hence we may assume that P and Q do not intersect in a vertex in W. Now $\big(W \setminus (\{w_1, \dots, w_{l-1}\} \cup \{w'_1, \dots, w'_{k-1}\})\big) \cup \{b_2, \dots, b_l\} \cup \{b'_2, \dots, b'_{k-1}\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 5 *If C is a component of G_0, then there are no two vertices w and w' of C that belong to W and two edges e and e' that belong to $E(G) \setminus E(G_0)$ such that w is incident with e, w' is incident with e', and e' is distinct from ww'.*

Proof: For contradiction, we assume that w, w', e, and e' are as specified. Let $e = wu$ and $e' = w'u'$. Note that $u' \neq w$.

If G_0 is disconnected, then Lemma 2 and Lemma 3 imply that u and u' belong to components of G_0 that are distinct from C, which implies a contradiction to Lemma 1 (i). Hence G_0 is connected.

Let $P : w_1b_1 \dots b_{l-1}w_l$ be a shortest path in C between $w = w_1$ and $w' = w_l$. If $u = w'$, then $W \setminus \{w'\}$ is a hull set of G, which is a contradiction. Hence $u \neq w'$, that is, e is distinct from ww'. If either $uw' \notin E(G_0)$ or $u'w \notin E(G_0)$, then $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. Hence $uw', u'w \in E(G_0)$, which implies that u and u' are distinct and belong to B. Now $W \setminus \{w'\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 6 *If C is a component of G_0, then there are no two edges wb and wb' that belong to $E(G) \setminus E(G_0)$ with $w \in W \cap V(C)$ and $b, b' \in B \cap V(C)$.*

Proof: For contradiction, we assume that such edges wb and wb' are as specified. Now $W \setminus \{w\}$ is a hull set of G, which is a contradiction. $\square$

Lemma 7 *If G_0 is connected and G is triangle-free, then there are no two edges ww' and bb' in G with $w, w' \in W$ and $b, b' \in B$.*

Proof: For contradiction, we assume that such edges ww' and bb' exist. Let $P : w_1b_1 \dots w_lb_l$ be a shortest path in G_0 between a vertex w_1 in $\{w, w'\}$ and a vertex b_l in $\{b, b'\}$. By symmetry, we may assume that $w_1 = w$ and $b_l = b$. Since G is triangle-free, b' is not adjacent to w_l. Now $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. $\Box$

Lemma 8 *If G_0 is connected and G is triangle-free, then there are no two edges wb and $b'b''$ in G with $w \in W$, $b, b', b'' \in B$, and $wb \in E_1$.*

Proof: For contradiction, we assume that such edges wb and $b'b''$ exist.

First, we assume that $b = b'$. Since G is triangle-free, w is not adjacent to b''. Let $P : w_1b_1 \dots w_lb_l$ be a shortest path in G_0 between $w = w_1$ and a vertex b_l in $\{b, b''\}$. Since G is triangle-free, w_l has only one neighbor in $\{b, b''\}$. By the choice of P, the unique vertex in $\{b, b''\} \setminus \{b_l\}$ has no neighbor in G_0 belonging to P. Now $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

Next, we assume that b is distinct from b' and b''. Let P be a shortest path in G_0 between a vertex in $\{w, b\}$ and a vertex in $\{b', b''\}$. If P is of the form $w_1b_1 \dots w_lb_l$ with $w_1 = w$, then $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_1, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. If P is of the form $b_1w_2 \dots w_lb_l$ with $b_1 = b$, then $(W \setminus \{w_2, \dots, w_l\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. $\Box$

Lemma 9 *If G_0 is connected and G is triangle-free, then there are no two distinct edges bb' and $b''b'''$ in G with $b, b', b'', b''' \in B$.*

Proof: For contradiction, we assume that such edges bb' and $b''b'''$ exist.

First, we assume that all four vertices b, b', b'', and b''' are distinct. If $P : b_1w_1 \dots w_{l-1}b_l$ is a shortest path in G_0 between a vertex b_1 in $\{b, b'\}$ and a vertex b_l in $\{b'', b'''\}$, then $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction.

Next we assume that $b' = b'''$, that is, the two edges bb' and $b''b'''$ are incident. Let $P : b_1w_1 \dots w_{l-1}b_l$ be a shortest path in G_0 between $b = b_1$ and $b'' = b_l$. Since G is triangle-free, b' is not adjacent to w_1 or w_{l-1}. Regardless of whether b' belongs to P or not, the set $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of G, which is a contradiction. $\Box$

3 Constructing all triangle-free graphs in $\mathcal{H}$

Let $\mathcal{G}_0$ denote the set of all bipartite graphs G_0 with a fixed bipartition $V(G_0) = B \cup W$ such that every vertex in B has degree exactly 2.

We consider four distinct operations that can be applied to a graph G_0 from $\mathcal{G}_0$.

- **Operation $\mathcal{O}_1$**

  Add one arbitrary edge to G_0.

- **Operation $\mathcal{O}_1'$**

  Select two vertices w_1 and w_2 from W and arbitrarily add new edges between vertices in $\{w_1, w_2\} \cup (N_{G_0}(w_1) \cap N_{G_0}(w_2))$.

- **Operation $\mathcal{O}_2$**

  Add one arbitrary edge between vertices in distinct components of G_0.

- **Operation $\mathcal{O}_3$**

  Choose a non-empty subset X of B such that all vertices in X are cut vertices of G_0 and no two vertices in X lie in the same component of G_0. Add arbitrary edges between vertices in X so that X induces a connected subgraph of the resulting graph. For every component C of G_0 that does not contain a vertex from X, add one arbitrary edge between a vertex in C and a vertex in X.

Let $\mathcal{G}_1$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_1$ once to a connected graph G_0 in $\mathcal{G}_0$. Let $\mathcal{G}_1'$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_1'$ once to a connected graph G_0 in $\mathcal{G}_0$. Let $\mathcal{G}_2$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_2$ once to a graph G_0 in $\mathcal{G}_0$ that has exactly two components. Let $\mathcal{G}_3$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_3$ once to a graph G_0 in $\mathcal{G}_0$ that has at least three components. Note that $\mathcal{O}_3$ can only be applied if G_0 has at least one cut vertex that belongs to B.

Finally, let

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_1' \cup \mathcal{G}_2 \cup \mathcal{G}_3. \tag{2}$$

Since operation $\mathcal{O}_1'$ allows that no edges are added, the set $\mathcal{G}_1'$ contains all connected graphs in $\mathcal{G}_0$.

Theorem 10 *$\mathcal{G} \subseteq \mathcal{H}$.*

Proof: Let G be a graph in $\mathcal{G}$ that is obtained by applying some operation to a graph G_0 in $\mathcal{G}_0$. Let $V(G_0) = B \cup W$ be the fixed bipartition of G_0. Since every vertex in B has two neighbors in W, the partite set W is a geodetic set of G and therefore $g(G) \leq |W|$. By (1), it suffices to show that $h(G) \geq |W|$ to conclude the proof. For contradiction, we assume that U is a hull set of G with $|U| < |W|$. Let D be a hull proof for U in G.

The proof naturally splits into four cases.

Case 1 $G \in \mathcal{G}_1$.

Let $W_1 = W \setminus U$ and $B_0 = B \cap U$. Note that, by the above assumption, $|W_1| > |B_0| \geq 0$.

We claim that there is at most one vertex w in W_1 for which the set $N_{G_0}(w)$ contains exactly one vertex of B_0 and that for every other vertex w' in W_1, the set $N_{G_0}(w')$ contains at least two vertices of B_0. In other words, there is a vertex w^* in W_1 such that

$$|N_{G_0}(w) \cap B_0| \geq \begin{cases} 1, & \text{if } w = w^*, \\ 2, & \text{if } w \in W_1 \setminus \{w^*\}. \end{cases} \tag{3}$$

Let w be a vertex in W_1. Since $|W_1| > |B_0|$, we may assume that $N_{G_0}(w)$ contains at most one vertex from B_0. Let x and y denote the two in-neighbors of w in D. Let e denote the edge added by operation $\mathcal{O}_1$.

If x belongs to W, then e is the edge xw. Hence $y \in B$ and $d_G(y) = 2$. Therefore $y \in B_0$, that is, $y \in N_{G_0}(w) \cap B_0$. Furthermore, for every other vertex w' in $W_1 \setminus \{w\}$, its two in-neighbors

x' and y' in D both belong to B and are not incident with e. Hence $d_G(x') = d_G(y') = 2$ and therefore $x', y' \in B_0$, that is, $x', y' \in N_{G_0}(w) \cap B_0$. Thus the claim holds. Hence, we may assume that x and y both belong to B.

If e is the edge wx, then we obtain as above that $y \in N_{G_0}(w) \cap B_0$. Hence x does not belong to B_0. This implies that the two edges of G_0 incident with x are both oriented towards x in D. For every other vertex w' in $W_1 \setminus \{w\}$, it follows that its two in-neighbors x' and y' in D satisfy $x', y' \in N_{G_0}(w) \cap B_0$. Hence, we may assume that e is neither wx nor wy.

Since $N_{G_0}(w)$ contains at most one element from B_0, we may assume that $x \in B_1$. Since $x \notin U$, x has two in-neighbors in D. This implies that the degree of x in G is at least 3. Hence e is incident with x and oriented towards x in D. This implies that $y \in N_{G_0}(w) \cap B_0$. Furthermore, for every other vertex w' in $W_1 \setminus \{w\}$, its two in-neighbors x' and y' in D both belong to B, and, if they are incident with e, then e is not oriented towards them in D. This implies that x' and y' belong to B_0, that is, $x', y' \in N_{G_0}(w') \cap B_0$.

Altogether, the existence of a vertex w^* in W_1 with (3) follows. If m denotes the number of edges in G_0 between W_1 and B_0, then (3) implies $m \geq 2(|W_1| - 1) + 1$. Furthermore, every vertex in B has degree 2 in G_0 and therefore $m \leq 2|B_0|$. Thus, $2|W_1| - 1 \leq 2|B_0|$. Since both cardinalities are integers, we obtain $|W_1| \leq |B_0|$, hence

$$|U| = |W \cap U| + |B \cap U| \geq |W \cap U| + |W \setminus U| = |W|,$$

which is a contradiction.

Case 2 $G \in \mathcal{G}_1'$.

Let w_1 and w_2 denote the two vertices from W selected by operation $\mathcal{O}_1'$ and let E denote the set of new edges added by operation $\mathcal{O}_1'$ between vertices in $\{w_1, w_2\} \cup (N_{G_0}(w_1) \cap N_{G_0}(w_2))$. Let $N = N_{G_0}(w_1) \cap N_{G_0}(w_2)$. Let $W_1 = W \setminus U$ and $B_0 = B \cap U$.

Again we claim that there is a vertex w^* in W_1 such that (3) holds. Let w be a vertex in W_1. As before we may assume that $N_{G_0}(w)$ contains at most one vertex from B_0. Let x and y denote the two in-neighbors of w in D.

First, we assume that x belongs to W. This implies that, by symmetry, we may assume that w is w_1, x is w_2, and the edge xw_1 belongs to E and is oriented towards w_1 in D. Since D is acyclic, the set N contains a vertex z such that no edge in E is oriented towards z in D, that is, z is a source of the digraph induced by N in D. This implies that z cannot have in-degree 2 in D, so $z \in N_{G_0}(w) \cap B_0$. If $w_2 \in W_1$, then N necessarily contains two vertices from B_0, which contradicts the assumption that $N_{G_0}(w)$ contains at most one vertex from B_0. Hence $w_2 \notin W_1$. Now for every other vertex w' in $W_1 \setminus \{w_1\}$, the two in-neighbors x' and y' of w' in D belong to B and are not incident with an edge from E. Therefore, $d_G(x') = d_G(y') = 2$, which implies $x', y' \in N_{G_0}(w') \cap B_0$.

Next, we assume that x and y both belong to B. By symmetry, we may assume that some edge in N is directed towards x in D. Hence $x \in N$, which implies that, by symmetry, we may assume that w is w_1. If z is a source of the digraph induced by N in D, then $z \in N_{G_0}(w) \cap B_0$. Since $N_{G_0}(w) \cap B_0$ contains at most one vertex, z is the unique vertex in $N_{G_0}(w) \cap B_0$. Hence $w_2 \in U$. Furthermore, for every other vertex w' in $W_1 \setminus \{w_1\}$, the two in-neighbors x' and y' of w' in D belong to B and are not incident with an edge from E. Therefore, $d_G(x') = d_G(y') = 2$, which implies $x', y' \in N_{G_0}(w') \cap B_0$.

Thus, the existence of a vertex w^* in W_1 with (3) follows, and double counting the number of edges between W_1 and B_0 yields the very same contradiction as at the end of Case 1.

Case 3 $G \in \mathcal{G}_2$.

Let H_1 and H_2 denote the two components of G_0. Let $[k] = \{1, 2, \ldots, k\}$. Let $k = 2$. For $i \in [k]$, let $W_1^i = V(H_i) \cap (W \setminus U)$ and $B_0^i = V(H_i) \cap (B \cap U)$. Recall that G arises from G_0 by adding exactly one edge between a vertex of H_1 and a vertex of H_2. The same arguments as in Case 1 imply that for $i \in [k]$, either W_1^i is empty or there is a vertex $w^{i,*}$ in W_1^i with

$$|N_{G_0}(w) \cap B_0^i| \geq \begin{cases} 1, & w = w^{i,*}, \\ 2, & w \in W_1^i \setminus \{w^{i,*}\}. \end{cases}$$

Double counting the edges between W_1^i and B_0^i as above yields $|W_1^i| \leq |B_0^i|$. Now

$$\begin{aligned} |U| &\geq \sum_{i \in [k]} (|V(H_i) \cap (W \cap U)| + |V(H_i) \cap (B \cap U)|) \\ &= \sum_{i \in [k]} \left(|V(H_i) \cap (W \cap U)| + |B_0^i|\right) \\ &\geq \sum_{i \in [k]} \left(|V(H_i) \cap (W \cap U)| + |W_1^i|\right) \\ &= \sum_{i \in [k]} (|V(H_i) \cap (W \cap U)| + |V(H_i) \cap (W \setminus U)|) \\ &= |W|, \end{aligned}$$

which is a contradiction.

Case 4 $G \in \mathcal{G}_3$.

Let X denote the set of vertices selected by operation $\mathcal{O}_3$. Let $H_1, \ldots, H_k$ denote the components of the graph that arises from G by deleting all vertices in X. By construction, there is exactly one edge between X and $V(H_i)$ for $i \in [k]$. For $i \in [k]$, let $W_1^i = V(H_i) \cap (W \setminus U)$ and $B_0^i = V(H_i) \cap (B \cap U)$. The same arguments as in Case 1 imply that for $i \in [k]$, either W_1^i is empty or there is a vertex $w^{i,*}$ in W_1^i with

$$|N_{G_0}(w) \cap B_0^i| \geq \begin{cases} 1, & w = w^{i,*}, \\ 2, & w \in W_1^i \setminus \{w^{i,*}\}. \end{cases}$$

Now the same counting argument as in Case 3 yields a contradiction. $\Box$

Unfortunately, the inclusion in Theorem 10 is strict (a graph in $\mathcal{H} \setminus \mathcal{G}$ is given in Figure 1). Nevertheless, in conjunction, the results in Sections 2 and Theorem 10 allow for a complete constructive characterization of the triangle-free graphs in $\mathcal{H}$. It is folklore that the geodetic number is NP-hard for triangle-free graphs.

Corollary 11 *If $\mathcal{T}$ denotes the set of all triangle-free graphs, then $\mathcal{G} \cap \mathcal{T} = \mathcal{H} \cap \mathcal{T}$.*

Proof: Theorem 10 implies $\mathcal{G} \cap \mathcal{T} \subseteq \mathcal{H} \cap \mathcal{T}$. For the converse inclusion, let G be a triangle-free graph in $\mathcal{H}$. Similarly as in Section 2, let W be a minimum geodetic set of G, let $B = V(G) \setminus W$, and let G_0 be a spanning bipartite subgraph of G with bipartition $V(G_0) = W \cup B$ such that every vertex in B has degree exactly 2 in G_0. Let E_1 denote the set of edges in $E(G) \setminus E(G_0)$

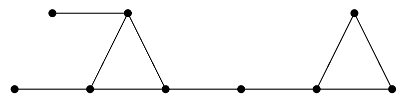

Figure 1: Example of graph in $\mathcal{H} \setminus \mathcal{G}$.

between vertices in the same component of G_0 and let E_2 denote the set of edges in $E(G)\setminus E(G_0)$ between vertices in distinct components of G_0.

First, we assume that G_0 is connected. In this case, $E_1 = E(G) \setminus E(G_0)$. For contradiction, we assume that E_1 contains two edges e and e'. By Lemmas 5 and 6, the edges e and e' are not both incident with vertices in W. We may therefore assume that e connects two vertices from B. Now, since G is triangle-free, Lemmas 7, 8, and 9 imply a contradiction. Hence E_1 contains at most one edge, which implies $G \in \mathcal{G}_1 \cup \mathcal{G}_1'$.

Next, we assume that G_0 is disconnected. By Lemmas 2 and 3, all vertices incident with edges in E_1 belong to B. For contradiction, we assume that E_1 is not empty. Let $bb' \in E_1$, where b and b' belong to some component C of G_0. Since G is connected but G_0 is not, some vertex of C is incident with an edge f from E_2. By Lemma 4, the edge f is not incident with b or b'. Furthermore, by Lemma 4 (ii) and (iii), G necessarily contains a triangle, which is a contradiction. Hence E_1 is empty. Now Lemma 1 immediately implies $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, which completes the proof. $\square$

Corollary 11 implies several restrictions on the cycle structure of a triangle-free graph G in $\mathcal{H}$. Let G_0 with bipartition $B \cup W$ be the underlying graph in $\mathcal{G}_0$. Clearly, all cycles of G that are also cycles of G_0 are of even length and alternate between B and W. Furthermore, at most one of the vertices from B in such a cycle can have degree more than 2 in G. If G_0 is connected, the cycles of G are either such cycles of G_0 or they contain the unique edges in $E(G) \setminus E(G_0)$. If G_0 has two components, then G arises from G_0 by adding a bridge and all cycles of G are also cycles of G_0. Finally, if G_0 has at least three components and X is as described in $\mathcal{O}_3$, then X induces an arbitrary connected triangle-free graph in G, that is, the cycle structure of $G[X]$ can be quite complicated. Nevertheless, all cycles in $G[X]$ contain only vertices of degree at least 4 in G. All further cycles of G are totally contained within one component of G_0 and contain at least one vertex from B that has degree 2 in G.

4 Recognizing all triangle-free graphs in $\mathcal{H}$

By Corollary 11, the structure of the triangle-free graphs in $\mathcal{H}$ is quite restricted. In fact, it is not difficult to recognize these graphs in polynomial time. This section is devoted to the details of a corresponding algorithm.

Let G be a given connected triangle-free input graph. By Corollary 11, the graph G belongs to $\mathcal{H}$ if and only if either G belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ or G belongs to $\mathcal{G}_3$.

Lemma 12 *It can be checked in polynomial time whether $G \in \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$.*

Proof: By definition, the graph G belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ if and only if deleting at most one edge from G results in a graph in $\mathcal{G}_0$ with at most two components. Since the graphs in $\mathcal{G}_0$ can obviously be recognized in linear time, it suffices to check whether $G \in \mathcal{G}_0$ and to consider each edge e of G in turn and check whether $G - e \in \mathcal{G}_0$. Since the graphs in $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ have a linear number of edges, all this can be done in quadratic time. $\square$

In view of Lemma 12, we may assume from now on that G does not belong $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. The following lemma is an immediate consequence of the definition of operation $\mathcal{O}_3$.

Lemma 13 *If G belongs to $\mathcal{G}_3$, then there is a vertex x of G of degree at least three and two edges $e_l = xy_l$ and $e_r = xy_r$ of G incident with x such that, in the graph G' that arises by deleting from G all edges incident with x except for e_l and e_r, the component $C(x, e_l, e_r)$ of G' that contains x has the following properties:*

(i) x is a cut vertex of $C(x, e_l, e_r)$;

(ii) $C(x, e_l, e_r)$ has a unique bipartition with partite sets $B_l \cup \{x\} \cup B_r$ and $W_l \cup W_r$;

(iii) Every vertex in $B_l \cup \{x\} \cup B_r$ has degree 2 in $C(x, e_l, e_r)$;

(iv) $B_l \cup W_l$ and $B_r \cup W_r$ are the vertex sets of the two components of $C(x, e_l, e_r) - x$ such that $y_l \in W_l$ and $y_r \in W_r$;

(v) None of the deleted edges connects x to a vertex from $V(C(x, e_l, e_r)) \setminus \{x\}$;

(vi) W_l and W_r both contain a vertex of odd degree.

Proof: Choosing as x one of the vertices from the non-empty set X in the definition of $\mathcal{O}_3$ and choosing as e_1 and e_2 the two edges of G_0 incident with x, the properties (i) to (v) follow immediately. Note that $C(x, e_l, e_r)$ is the component of G_0 that contains x. For property (vi), observe that the number of edges of $C(x, e_l, e_r)$ between $B_l \cup \{x\}$ and W_l is exactly $2|B_l| + 1$, that is, it is an odd number, which implies that not all vertices of W_l can be of even degree. A similar argument applies to W_r. $\square$

The key observation for the completion of the algorithm is the following lemma, which states that the properties from Lemma 13 uniquely characterize the elements of X.

Lemma 14 *If G belongs to $\mathcal{G}_3$ and a vertex x of G of degree at least three and two edges $e_l = xy_l$ and $e_r = xy_r$ of G incident with x are such that properties (i) to (vi) from Lemma 13 hold, then*

(a) G is obtained by applying operation $\mathcal{O}_3$ to a graph G_0 in $\mathcal{G}_0$ with at least three components such that x belongs to the set X used by operation $\mathcal{O}_3$ and

(b) $C(x, e_l, e_r)$ defined as in Lemma 13 is the component of G_0 that contains x.

Proof: Let G arise by applying operation $\mathcal{O}_3$ to a graph G_0 in $\mathcal{G}_0$ with at least three components and let X be the corresponding set used by operation $\mathcal{O}_3$. Let $V(G_0) = B \cup W$ be the underlying bipartition of G_0. Clearly, by possibly increasing X, we may assume that every vertex b in B that is incident with an edge in $E(G) \setminus E(G_0)$ either belongs to X or is no cut vertex of the component of G_0 it belongs to.

Let a vertex x of G of degree at least three and let two edges $e_l = xy_l$ and $e_r = xy_r$ of G incident with x be such that properties (i) to (vi) from Lemma 13 hold. Let G' arise by deleting from G all edges incident with x except for e_l and e_r, let C denote the component of G_0 that contains x, and let $C(x, e_l, e_r)$ be the component of G' that contains x.

We will prove that x belongs to X, that is, that (a) holds. Furthermore, we will prove that the desired second statement (b) holds either for G_0 or for a slightly modified alternative choice of G_0.

For contradiction, we assume that x does not belong to X.

First, we assume that $x \in B \setminus X$. Since x has degree at least three in G and has degree two in G_0, the vertex x is incident with an edge xy in $E(G) \setminus E(G_0)$. If $y \in \{y_l, y_r\}$, then, by operation $\mathcal{O}_3$, the vertex y belongs to X. If C' denotes the component of G_0 that contains y, then a similar argument as in the proof of Lemma 13 implies that $W \cap V(C')$ contains a vertex of odd degree. Therefore, $C(x, e_l, e_r)$ contains a vertex of odd degree that is within even distance from x, which contradicts properties (ii) and (iii). Hence $y \notin \{y_l, y_r\}$, which implies that $C(x, e_l, e_r)$ coincides with C. Now x is no cut vertex of C, contradicting property (i). Hence x cannot belong to $B \setminus X$.

Next, we assume that $x \in W$. By operation $\mathcal{O}_3$, the vertex x can be incident with at most one edge from $E(G) \setminus E(G_0)$. Furthermore, if x is incident with such an edge, then no other vertex in C is incident with an edge from $E(G) \setminus E(G_0)$. Therefore, by symmetry, we may assume that no vertex in $B_l \cup W_l$ defined as in Lemma 13 is incident with an edge from $E(G) \setminus E(G_0)$. This implies that $B_l \cup W_l$ is a subset of $V(C)$. Since the vertices in W_l are all at odd distance from x in $C(x, e_l, e_r)$, they all belong to $B \cap V(C)$. By property (v), all these vertices are of degree two in $C(x, e_l, e_r)$, which implies a contradiction to property (vi). Hence x cannot belong to W.

It follows that x belongs to X, that is, (a) holds.

We proceed to the proof of (b). As already noted above, we prove that (b) holds either for G_0 or for a slightly modified alternative choice of G_0. If (b) does not hold for G_0, that is, $C(x, e_l, e_r)$ is distinct from C, then the edges e_l and e_r do not coincide with the two edges of G_0 that are incident with x, say $e_l = xy_l \notin E(G_0)$. Now y_l is the unique vertex of some component C' of G_0 that is incident with an edge in $E(G) \setminus E(G_0)$. It follows as above that y_l cannot belong to X. Hence y_l belongs to $(B \setminus X) \cup W$. If $y_l \in W$, then we can exchange the component C' with one of the two components of $C - x$ within G_0 and (b) follows for this modified G_0. If $y_l \in B \setminus X$, then the properties (i) to (vi) imply that C' is a 2-regular bipartite graph, that is, C' is an even cycle. In this case, we can swap the bipartition of C' and perform a similar exchange within G_0 as in the case $y_l \in W$. Therefore, also in this case, (b) follows for this modified G_0. $\square$

We proceed to the main result in this section.

Theorem 15 *For a given triangle-free graph G, it can be checked in polynomial time whether $h(G) = g(G)$ holds.*

Proof: Clearly, we can consider each component of G separately and may therefore assume that G is connected. Let n denote the order of G. By Lemma 12, we can check in $O(n^2)$ time whether G belongs $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. If this is the case, then Corollary 11 implies $h(G) = g(G)$. Hence, we may assume that G does not belong to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. Note that there are $O(n^3)$ choices for a vertex x of G and two incident edges e_l and e_r of G. Furthermore, note that for every individual choice of the triple (x, e_l, e_r), the properties (i) to (vi) from Lemma 13 can be checked in $O(n)$ time. Therefore, by Lemmas 13 and 14, in $O(n^4)$ time, we can

- either determine that no choice of (x, e_l, e_r) satisfies the conclusion of Lemma 13, which, by Corollary 11, implies $h(G) \neq g(G)$,
- or find a suitable triple (x, e_l, e_r) and reduce the instance G to a smaller instance $G^- = G - V(C(x, e_l, e_r))$.

Since the order of G^- is at least three less than n, this leads to an overall running time of $O(n^5)$. $\square$

The algorithm outlined in the proof of Theorem 15 allows to determine for every triangle-free graph G with $h(G) = g(G)$, a bipartite subgraph G_0 with bipartition $V(G_0) = W \cup B$ as in Section 2. Since $g(G) = g(G_0) = |W|$, it is therefore possible to determine $g(G)$ in polynomial time for these graphs.

5 Forbidden induced subgraphs

In this last section, we give a complete characterization of the maximal hereditary subclass of $\mathcal{H}$ in terms of forbidden induced subgraphs.

It is an easy exercise to prove $h(G) = g(G)$ whenever G is a path, a cycle, or a star.

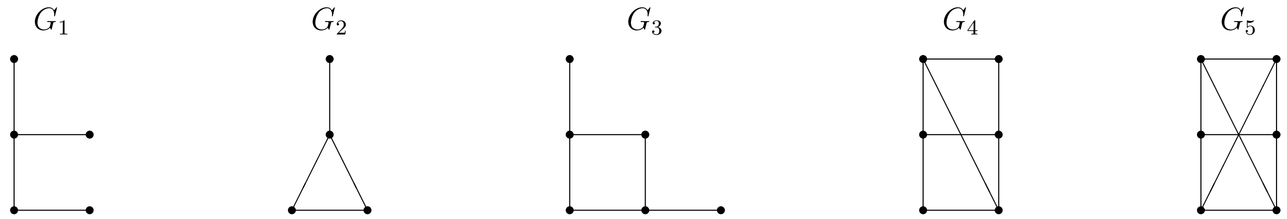

Figure 2: The five forbidden subgraphs $G_1, \dots, G_5$.

Let $G_1, \dots, G_5$ be the graphs depicted in Figure 2.

Theorem 16 *If G is a graph, then $h(G') = g(G')$ for every induced subgraph G of G if and only if G is $\{G_1, \dots, G_5\}$-free.*

Proof: Since $3 = h(G_1) = h(G_3) < g(G_1) = g(G_3) = 4$ and $2 = h(G_2) = h(G_4) = h(G_5) < g(G_2) = g(G_4) = g(G_5) = 3$, the "only if"-part of the statement follows. In order to prove the "if"-part, we may assume, for contradiction, that G is a connected $\{G_1, \dots, G_5\}$-free graph with $h(G) < g(G)$. We consider different cases.

Case 1 *G contains a triangle $T : abca$.*

Since G is G_2-free, no vertex has exactly one neighbor in T.

If some vertex has no neighbor in T, then, by symmetry, we may assume that there are two vertices u and v of G such that uva is a path and u has no neighbor in T. Since G is G_2-free, we may assume that v is adjacent to b. Now u, v, a, and b induce G_2, which is a contradiction.

Hence every vertex has at least one neighbor in T. This implies that the vertex set of G can be partitioned as

$$V(G) = \{a,b,c\} \cup N(\{a,b\}) \cup N(\{a,c\}) \cup N(\{b,c\}) \cup N(\{a,b,c\}),$$

where $N(S) = \{u \in V(G) \setminus \{a,b,c\} \mid N_G(u) \cap \{a,b,c\} = S\}$.

If two vertices, say u and v, in $N(\{a,b\})$ are adjacent, then u, v, a, and c induce G_2, which is a contradiction. Hence, by symmetry, each of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$ is independent. If some vertex u in $N(\{a,b\})$ is not adjacent to some vertex v in $N(\{a,c\})$, then u, v, a, and b induce G_2, which is a contradiction. Hence, by symmetry, there are all possible edges between every two of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$. If some vertex u in $N(\{a,b\})$ is not adjacent to some vertex v in $N(\{a,b,c\})$, then u, v, a, and c induce G_2, which is a contradiction. Hence, by symmetry, there are all possible edges between the two sets $N(\{a,b\}) \cup N(\{a,c\}) \cup N(\{b,c\})$ and $N(\{a,b,c\})$. If $N(\{a,b\})$ contains exactly one vertex, say u, then $I_G(\{u,c\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$. Hence, by symmetry, none of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$ contains exactly one vertex. If there are two vertices in $N(\{a,b\})$, say u_1 and u_2, and two vertices in $N(\{b,c\})$, say v_1 and v_2, then u_1, u_2, v_1, v_2, a, and c induce G_5, which is a contradiction. Hence no two of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$ contain at least two vertices.

Altogether, we may assume, by symmetry, that $N(\{a,c\})$ and $N(\{b,c\})$ are empty. Now $I_G(\{a,b\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$ and completes the proof in this case.

Case 2 *G contains no triangle but a cycle C of length four: $abcda$.*

If some vertex has no neighbor in C, then, by symmetry, we may assume that there are two vertices u and v of G such that uva is a path. Since G is triangle-free, v is not adjacent to b or d. Hence u, v, a, b, and d induce G_1, which is a contradiction. Hence every vertex has at least one neighbor in C. Since G is triangle-free, this implies that the vertex set of G can be partitioned as

$$V(G) = \{a,b,c,d\} \cup N(\{a\}) \cup N(\{b\}) \cup N(\{c\}) \cup N(\{d\}) \cup N(\{a,c\}) \cup N(\{b,d\}),$$

where $N(S) = \{u \in V(G) \setminus \{a,b,c,d\} \mid N_G(u) \cap \{a,b,c,d\} = S\}$.

If there is a vertex u in $N(\{a\})$ and a vertex v in $N(\{c\})$, then u and v are adjacent, because G is G_3-free. Now u, v, a, b, and d induce G_1, which is a contradiction. Hence, by symmetry, we may assume that $N(\{c\}) \cup N(\{d\})$ is empty. If there is a vertex u in $N(\{b\})$ and a vertex v in $N(\{a,c\})$, then u and v are not adjacent, because G is G_4-free. Now u, v, a, b, and d induce G_1, which is a contradiction. Hence, by symmetry, one of the two sets $N(\{b\})$ and $N(\{a,c\})$ is empty and one of the two sets $N(\{a\})$ and $N(\{b,d\})$ is empty. If there is a vertex u in $N(\{a,c\})$ and a vertex v in $N(\{b,d\})$, then u, v, a, b, c, and d induce either G_4 or G_5, which is a contradiction. Hence, by symmetry, we may assume that $N(\{b,d\})$ is empty.

Since G is G_1-free, there are all possible edges between the two sets $N(\{a\})$ and $N(\{b\})$.

Since G is G_1-free, both of the sets $N(\{a\})$ and $N(\{b\})$ contain at most one vertex.

Since G is triangle-free, there is no edge between $N(\{a\})$ and $N(\{a,c\})$.

If both of the sets $N(\{a\})$ and $N(\{b\})$ are not empty, then G is a graph of order 6 with $h(G) = g(G) = 3$, which is a contradiction. Hence, by symmetry, we may assume that

$$V(G) = \{a,b,c,d\} \cup N(\{a\}) \cup N(\{a,c\}).$$

If $N(\{a\})$ is empty, then $I_G(\{a,c\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$. Hence $N(\{a\})$ contains exactly one vertex, say u, and $I_G(\{a,c,u\}) = V(G)$, which implies $g(G) \leq 3$. If $H_G(U) = V(G)$ for some set U of vertices of G, then $u \in U$. In view of the structure of G, it follows easily that $h(G) \geq 3$, which implies the contradiction $h(G) = g(G)$ and completes the proof in this case.

Case 3 *G does not contain a triangle or a cycle of length four.*

If G contains no vertex of degree at least 3, then G is a path or a cycle, which implies the contradiction $h(G) = g(G)$. Hence, we may assume G contains a vertex of degree at least 3. Since G is G_1-free, G is a star, which implies the contradiction $h(G) = g(G)$ and completes the proof. □

6 Conclusion

This paper discloses a number of properties of the class $\mathcal{H}$ of graphs G for which $h(G) = g(G)$, allowing for the efficient recognition of two rather comprehensive subclasses of $\mathcal{H}$: the class of triangle-free graphs in $\mathcal{H}$, and the maximal hereditary subclass of $\mathcal{H}$.

As for open problems, at least two of them are immediate: to give a constructive characterization of all graphs in $\mathcal{H}$, and to describe an efficient algorithm to recognize all graphs in $\mathcal{H}$.

Acknowledgments. This work was supported in part by the DAAD/CAPES Probral project "*Cycles, Convexity and Searching in Graphs*" grant number 50724218.

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