# On the embedding of cone graphs in the line with distinct distances between neighbors

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**Abstract**

A graph $G = (V, E)$ is graceful if its vertices can be embedded in the interval $[0, |E|]$ in such a way that: (1) all vertices have integer coordinates; (2) no two vertices share the same coordinate; and (3) the distances in the line between two adjacent vertices in the graph are all distinct. We show that the generalized cone graph $C_p + I_q$ (the join of a cycle and an independent set) is graceful for $p \in \{9, 13, 17\}$ and $q \geq 1$. We also show that $C_p + I_q$ is not graceful for several odd values of $q$ with $p \equiv 2 \pmod 4$, disproving a conjecture of Brundage. Our results suggest a conjecture towards the characterization of graceful generalized cone graphs.

*Keywords:* distance geometry, computational proofs, generalized cone graphs, graceful labeling

## 1. Introduction

The fundamental problem in Distance Geometry is to find a set of points in a geometric space (typically the 3-dimensional Euclidean space) that conform to certain constraints given in terms of distances between pairs of points. In the classic version of the problem [16], the input is a graph in which each edge is

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labeled with the distance that its corresponding end vertices must have in the geometric space. However, Distance Geometry provides a broader framework to study classic graph-theoretical problems [14, 18]. Some examples follow.

- Recognizing whether a given graph is a unit disk graph (i.e., whether it is the intersection graph of congruent disks in the plane) is equivalent to asking for an embedding of the vertices in the plane such that two vertices are at distance at most 1 if, and only if, they are adjacent in the graph [5].
- The $k$-colorability problem takes as input a graph and an integer $k > 0$, and asks for an embedding of the vertices in the interval $[0, k]$ with integer coordinates such that the distance between vertices that are adjacent in the graph is strictly positive [7].
- The list coloring problem is similar to the $k$-colorability problem, except that a list of allowed coordinates is assigned to each vertex as part of the input [6].

Differently from the fundamental Distance Geometry Problem, in which precise distances are given, the input of the problems above comprises *sets* (either discrete or continuous) of acceptable distances between points corresponding to vertices that are adjacent in the graph.

In the present work, we consider the Graceful Labeling problem under the Distance Geometry standpoint. The input is a graph $G = (V, E)$, and the goal is to find an embedding of its vertices in the interval $[0, |E|]$ such that:

- all vertices have integer coordinates;
- no two vertices are assigned the same coordinate; and
- the distances between vertices that are adjacent in the graph are all distinct.

Note that graph labeling is a discrete problem in the sense that the space of possible solutions is finite. It resembles the famous Discretizable Molecular

Distance Geometry Problem [15] (DMDGP), which is a particular case of the Distance Geometry Problem formulated as a search in a discrete space. More precisely, a DMDGP instance consists of a graph whose vertices can be ordered in such a way that any four consecutive vertices induce a complete graph—hence, the possible 3-dimensional positions of each vertex consist of only two positions. The discrete aspect of the problem suggests a computational approach to search for solutions, such as the branch-and-prune algorithm by Lavor et al. [15]. While the graph labeling problem is somehow analogous to the DMDGP, computational algorithms for graph labeling have to deal with the fact that the solution space is much larger, since the number of possible positions for each vertex is $O(|E|)$, not $O(1)$ as in the DMDGP. In the present work, we also follow a computational approach based on a branch-and-prune algorithm to implicitly enumerate all the possible solutions of an instance, allowing not only to determine if a graph is graceful but also to count the number of solutions and to determine relevant properties of the solutions.

Graceful labeling was originally introduced in 1966 by Rosa [20]. The famous Graceful Tree Conjecture states that every tree has a graceful labeling. The validity of the conjecture would immediately imply another conjecture by Ringel: for any tree $T$ with $n$ vertices, the complete graph $K_{2n-1}$ can be decomposed into $2n-1$ trees isomorphic to $T$. The relevance and difficulty of the graceful labeling problem is widely recognized in the complexity and combinatorics communities. Indeed, the Graceful Graph Problem, i.e., deciding whether a graph is graceful or not, appeared as the problem of the month in David S. Johnson's NP-completeness Column of 1983 [12] and it was revived as a long-standing open problem in the NP-completeness Column of 2005 [13]. After fifty years of its statement, the Graceful Tree Conjecture and the Graceful Graph Problem remain open, despite the large number of papers devoted to the subject.

Besides their theoretical impact and scientific challenge, graceful labelings have numerous applications in diverse fields. In 1977, Bloom [2] described a series of applications of graceful labelings to code design, X-ray crystallographic analysis, circuit layouts design, and communication network addressing.

Considering the theoretical impact, the present and the potential applications of graph labelings, it is not surprising that much research is devoted to solve the recognition problem at least partially—in particular, investigating the Graceful Graph Problem on restricted graph classes. Here, we consider the *generalized cone graphs*, a graph class defined by the join $C_p + I_q$ of a cycle graph $C_p$ and an independent set $I_q$, with $p \geq 3$ and $q \geq 0$.

For the sake of clarity, we provide next the classic definition of graceful labeling. Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. A *graceful labeling* of $G$ is an injective vertex labeling $f \colon V \to \{0, 1, \ldots, m\}$ such that the resulting edge labeling $f_\gamma \colon E \to \{1, 2, \ldots, m\}$ with $f_\gamma(uv) = |f(u) - f(v)|$ is also injective. A graph that admits a graceful labeling is called a graceful graph (see Figure 1).

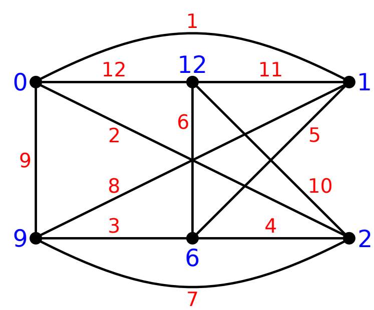


Figure 1: A graceful labeling of $C_4 + I_2$.

Throughout the text, let $V(C_p + I_q) = \{u_0, u_1, \ldots, u_{p-1}, v_0, v_1, \ldots, v_{q-1}\}$, where $u_k$ is a vertex from the cycle $C_p$ and $v_k$ is a vertex from the independent set $I_q$. For simplicity, the set of integers $\{a, a+1, a+2, \ldots, b\}$ is denoted $[a, b]$. From now on, we call generalized cone graphs as cone graphs, or simply cones.

## 2. Preliminary results

We start with a well known fact about graceful labeling in general [11, 20].

**Lemma 1.** *All Eulerian graphs with* $m \equiv 1, 2 \pmod 4$ *are not graceful.*

Lemma 1 is known as the *parity condition.* For cycles, the parity condition actually characterizes gracefulness [20], i.e., the cycle $C_p$ is graceful if, and only if, $p \equiv 0, 3 \pmod 4$. For cone graphs, the following holds:

**Proposition 1.** *The cone graph $C_p + I_q$ is not graceful for $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 2$.*

The join of a cycle on $p$ vertices and a singleton graph is the wheel graph $W_p = C_p + I_1$. Frucht [9] showed that all wheel graphs are graceful. The graphs $C_p + I_2$ are known as *double cones.* It is known that many of them are graceful [1, 10, 19], but it is still unknown whether they are all graceful except for those eliminated by the parity condition.

For generalized cone graphs, Bhat-Nayak and Selvam [1] showed that $C_p+I_q$ is graceful for $p \equiv 0, 3 \pmod{12}$ and $q \geq 1$. They also proved that $C_p + I_q$ with $p \in \{4, 7, 11, 19\}$ are graceful for all $q \geq 1$, and they presented a graceful labeling for the double cones $C_5 + I_2$ and $C_9 + I_2$. Later on, Brundage [3] presented a graceful labeling for $C_6+I_3$ and showed that $C_5+I_q$ and $C_8+I_q$ are graceful for all $q \geq 1$. Next, we show Brundage's construction to obtain graceful labelings for these two families of cones.

**Proposition 2.** *The cone graphs $C_5+I_q$ and $C_8+I_q$ are graceful for all $q \geq 1$.*

*Proof.* Brundage gives a graceful labeling $f : V \to [0, m]$ for each case.

For $C_5 + I_q$, label the vertices of $C_5$ with $0, m, m-3, 3, m-1$ consecutively along the cycle, where $m = 5(q + 1)$ is the total number of edges. Now, label the vertices of $I_q$ as follows:

$$f(v_k) = \begin{cases} 2, & \text{if } k = 0; \\ 5k+3, & \text{if } k = 1, 2, \ldots, q-1. \end{cases}$$

Thus, for a $v_k$, $0 < k < q$, as $3 < 5k + 3 < m - 3$, its incident edges have labels $5k + 3, m - (5k + 3), (m - 3) - (5k + 3), 5k, (m - 1) - (5k + 3)$, which are all distinct since they have different residues modulo 5: $5k + 3 \equiv 3 \pmod 5, m - (5k + 3) \equiv 2 \pmod 5, (m - 3) - (5k + 3) \equiv 4 \pmod 5, 5k \equiv 0$

(mod 5), $(m-1)-(5k+3) \equiv 1 \pmod 5$. It is now easy to see that the labels in the edges incident with $v_k$, with $0 < k < q$, cover the whole interval $[4, m-7]$. Along with the labels of the edges in $C_5$ $(m, 3, m-6, m-4, m-1)$ and those incident with $v_0$ $(2, m-2, m-5, 1, m-3)$, all the labels in $[1, m]$ appear exactly once. Thus, $f$ is a graceful labeling of $C_5 + I_q$.

For $C_8 + I_q$, label the vertices of $C_8$ with $0, m, 2, 3, m-2, 1, m-3, m-1$ along the cycle, where $m = 8(q+1)$, and label each $v_k$ in $I_q$ with $4k+6$. The proof that this is indeed a graceful labeling is analogous to the previous case. □

*Remark* 1. Note that the graceful labeling for some families of cone graphs are often not unique. For instance, a graceful labeling for $C_8 + I_q$ distinct from the one given by Brundage goes as follows. Label $C_8$ with $0, m, \frac{m}{2}, \frac{3m}{4} + 1, \frac{m}{2} + 1, \frac{3m}{4}, \frac{m}{4} - 1, m-1$, and label $I_q$ with $2k+2$ for $0 \leq k < q$, where $m = 8(q+1)$.

## 3. New families of graceful and non-graceful cones

We present results concerning both graceful and non-graceful cone graphs.

**Proposition 3.** *The cone graphs $C_9 + I_q$, $C_{13} + I_q$, and $C_{17} + I_q$ are graceful for all $q \geq 1$.*

*Proof.* Label the independent set $I_q$ as follows, where $p$ is the size of the corresponding cycle:

$$f(v_k) = \begin{cases} 1, & \text{if } k = 0 \\ pk + 4, & \text{if } k = 1, 2, \ldots, q-1 \end{cases}$$

As for the cycle, label its vertices as follows along the cycle:

$C_9$: $0, m, 5, m-7, 3, m-8, m-3, 4, m-2$;

$C_{13}$: $0, m, m-8, 6, m-9, 10, m-6, 7, m-4, m-7, 5, m-1, m-3$;

$C_{17}$: $0, m, 7, m-8, m-3, 14, m-11, 9, m-13, 5, m-5, m-15, 12, m-7, m-10, 3, m-2$. □

The graceful labelings given in the proof of Proposition 3 suggest that every cone graph with $p \equiv 1 \pmod 4$, $p \geq 9$, admits a graceful labeling such that the independent set is labeled that way.

As a matter of fact, looking at the other classes of equivalences modulo 4, we found out a similar property. For $p \equiv 0 \pmod 4$, it seems that $C_p + I_q$ has a graceful labeling in which $f(v_k) = \frac{p}{4}(k+1)$ for $0 \leq k < q$. This can be verified for $p = 4$ [1] and $p = 8$ (see Remark 1). Similarly for $p \equiv 3 \pmod 4$, $p \geq 7$, the cone graph $C_p + I_q$ seems to have a graceful labeling with $f(v_0) = 2$ and $f(v_k) = pk + 4$ for $1 \leq k < q$.

As for $p \equiv 2 \pmod 4$, we have a completely different behavior, starting with even values of $q$, which are all non-graceful (see Proposition 1). Brundage [3] actually conjectured that the non-graceful cone graphs given by the parity condition would be the only non-graceful ones. However, we have disproved his conjecture by showing that $C_6 + I_5$, a non-Eulerian graph, is not graceful. Our findings go further by establishing that $C_6 + I_q$ is not graceful for $5 \leq q \leq 35$.

Furthermore, although $C_{10} + I_3$ and $C_{10} + I_5$ are graceful, $C_{10} + I_q$ is not graceful for $7 \leq q \leq 25$. Likewise, $C_{14}+I_3$ and $C_{14}+I_5$ are graceful, but $C_{14}+I_7$ and $C_{14} + I_9$ are not. The following two propositions summarize these findings.

**Proposition 4.** *The cone graphs $C_{10}+I_q$ and $C_{14}+I_q$ are graceful for $q = 3, 5$.*

*Proof.* We have the following labelings where the first $p$ labels are from the cycle and the last $q$ are from the independent set.

$C_{10} + I_3$: 0, 40, 25, 3, 33, 13, 6, 29, 10, 21; 37, 38, 39.

$C_{10} + I_5$: 0, 27, 1, 57, 14, 13, 2, 16, 3, 15; 32, 23, 51, 55, 60.

$C_{14} + I_3$: 0, 56, 6, 1, 28, 5, 2, 30, 34, 3, 33, 11, 22, 55; 40, 47, 54.

$C_{14}+I_5$: 0, 84, 33, 17, 82, 34, 47, 54, 64, 68, 69, 32, 49, 83; 14, 11, 8, 5, 2. □

**Proposition 5.** *The cone graphs $C_6 + I_q$, $5 \leq q \leq 35$, $C_{10} + I_q$, $7 \leq q \leq 25$, $C_{14} + I_7$, and $C_{14} + I_9$ are not graceful.*

The proof of Proposition 5 was achieved computationally.[1] We devised an

[1] The code can be found at `http://github.com/rodrigozhou/graceful-cone-graphs`.

implicit enumeration algorithm with adequate pruning, which managed to exhaust all possible labelings for the cone graphs in Proposition 5. None of them was found to be graceful.

Our backtracking-based approach is similar to the one employed by Fang [8]. At each iteration, label a yet unlabeled vertex so that a new edge label appears, making sure that edge labels are revealed in strictly descending order to reduce branching on the search trees.

Besides the "descending order of edge labels" idea, the symmetries inherent to cone graphs played a fundamental role in pruning the search tree, avoiding that our search went through equivalent/symmetric labelings repeatedly. Also, the *complementarity property* of graceful labelings (if every vertex label $\ell$ is replaced with $m - \ell$ in a graceful labeling, the resulting labeling is also graceful) was instrumental in reducing the computational time of our algorithm. Indeed, if our search algorithm were based on plain brute force, the combinatorial explosion of the search space, so common in backtracking-based explorations, would probably have made any useful outcome out of reach. Next, we list the observations included in the search algorithm, which allowed it to run in a reasonable amount of time for the considered instances.

*Force $f(u_0) = 0$ without loss of generality.* Because the number of available edge labels is equal to the number of edges, the edge labeling function $f_\gamma$ produced by a graceful labeling is not only injective but also an onto function. This means that, in a graceful labeling, every possible edge label from 1 to $m$ must appear as a label of some edge. Since an edge label is obtained as the absolute difference of the labels of its incident vertices, it follows that the vertices labeled 0 and $m$ must be adjacent in the graph (otherwise no edge would be assigned label $m$). Since, in a generalized cone graph, all edges are incident with at least one vertex of the cycle, one of the vertices of the cycle must be labeled 0 or $m$. By symmetry, let $u_0$ be that vertex. Now, the complementarity property allows us to assume without loss of generality that $f(u_0) = 0$.

*Establish an order of labeling in $I_q$.* Since all vertices in the independent set are indistinguishable between themselves (both from the standpoint of some vertex in the independent set, since there are no edges between any of them, and from the standpoint of some vertex in the cycle, since each vertex in the cycle is adjacent to all vertices in the independent set), we may assume an order in which the vertices of $I_q$ are labeled. This prevents looking for labelings that are identical up to a permutation of the vertices in $I_q$.

*Just two candidate recipients for vertex label $m$.* Assuming $f(u_0) = 0$, the vertex label $m$ must be assigned to a vertex that is adjacent to $u_0$, i.e., to either $u_1$, $u_{p-1}$ or $v_k$ for some $k \in [0, q-1]$. Again owing to the symmetries in both the cycle and the independent set, we can narrow down our options, without loss of generality, to only two among those vertices, say $u_1$ and $v_0$.

*Constrained recipients for edge label $m-1$.* If, in the previous step, we chose vertex $u_1$ to receive label $m$, then, because we had already assigned label 0 to vertex $u_0$, the edge label $m-1$ can only appear on an edge that is incident with either $u_1$ (a neighbor of $u_1$ would receive label 1) or $u_0$ (a neighbor of $u_0$ would receive label $m-1$). Owing to the symmetries (rotation, reflection) of the cycle and the complementarity property, these two cases are actually equivalent. We can therefore consider, without loss of generality, that the edge labeled $m-1$ will be incident with $u_1$. We must now pick a neighbor of vertex $u_1$ to assign label 1. Since vertex $u_0$ is already labeled 0, the possible neighbors are $u_2$ or $v_k$. However, by the symmetry of the independent set, we can consider $v_0$ as the sole candidate to receive label 1, and our search is limited to just two cases. If, on the other hand, we chose vertex $v_0$ to receive label $m$, then we must either assign label $m-1$ to a neighbor of $u_0$ (namely $u_1$ or $v_1$ without loss of generality), or assign label 1 to a neighbor of $v_0$ (namely $u_k$, where we can impose $1 \leq k \leq \lfloor \frac{p}{2} \rfloor$ owing to the reflection symmetry of the cycle).

## 4. Future directions

Table 1 describes the state of the art with regard to the gracefulness of generalized cone graphs. We summarize our contribution as follows:

- We spotted a number of non-graceful cones for which the parity condition does not apply, disproving a conjecture by Brundage.
- We exhibited graceful labelings for three infinite subfamilies of cone graphs, namely $C_9 + I_q$, $C_{13} + I_p$, and $C_{17} + I_q$, for all $q \geq 1$.
- We asserted the gracefulness of $C_{10} + I_3$, $C_{10} + I_5$, $C_{14} + I_3$, and $C_{14} + I_5$.

| $q$ \ $p$ | $3, 4$ | $5$ | $6$ | $7, 8$ | $9$ | $10$ | $11, 12$ | $13$ | $14$ | comments |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Y | N | N | Y | N | N | Y | N | N | Y iff $p \equiv 0, 3 \pmod 4$ |
| 1 | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y $\forall p$ |
| 2 | Y | Y | N | Y | Y | N | Y | Y | N | ?, N $\forall p = 6 + 4k$ |
| 3 | Y | Y | Y | Y | Y | Y | Y | Y | Y | ? |
| 4 | Y | Y | N | Y | Y | N | Y | Y | N | ?, N $\forall p = 6 + 4k$ |
| 5 | Y | Y | N | Y | Y | Y | Y | Y | Y | ? |
| 6 | Y | Y | N | Y | Y | N | Y | Y | N | ?, N $\forall p = 6 + 4k$ |
| 7 | Y | Y | N | Y | Y | N | Y | Y | N | ? |
| 8 | Y | Y | N | Y | Y | N | Y | Y | N | ?, N $\forall p = 6 + 4k$ |
| 9 | Y | Y | N | Y | Y | N | Y | Y | N | ? |
| 10 | Y | Y | N | Y | Y | N | Y | Y | N | ?, N $\forall p = 6 + 4k$ |
| 11 | Y | Y | N | Y | Y | N | Y | Y | ? | ? |
| comments | Y | Y $\forall q \geq 1$ | ?, N $\forall q$ even | Y | Y $\forall q \geq 1$ | ?, N $\forall q$ even | Y | Y $\forall q \geq 1$ | ?, N $\forall q$ even | ?, N $\forall p = 6 + 4k$, $q$ even; Y $\forall p \equiv 0, 3 \pmod{12}$ |

Table 1: Gracefulness of $C_p + I_q$ (shaded entries are new results).

Although our findings do not characterize the class of graceful generalized cone graphs, they do suggest directions on where to look for graceful labelings in the case of $p \not\equiv 2 \pmod 4$. They are summarized by Conjectures 1 and 2.

**Conjecture 1.** *For $p \not\equiv 2 \pmod 4$, $p \geq 7$, there exists a graceful labeling of $C_p + I_q$ such that:*

- *if $p \equiv 0 \pmod 4$, then $f(v_k) = \frac{p}{4}(k+1)$ for $0 \leq k < q$;*

- *if* $p \equiv 1 \pmod 4$, *then* $f(v_0) = 1$ *and* $f(v_k) = pk + 4$ *for* $1 \leq k < q$*;*
- *if* $p \equiv 3 \pmod 4$, *then* $f(v_0) = 2$ *and* $f(v_k) = pk + 4$ *for* $1 \leq k < q$*.*

**Conjecture 2.** *For every* $p \equiv 2 \pmod 4$*, there exists a* $q_p > 1$ *such that the cone graph* $C_p + I_q$ *is not graceful for all* $q \geq q_p$*.*

Another intriguing pattern we may report after having observed all existing graceful labelings for a fixed $C_p + I_q$ is that the graphs $C_9 + I_q$, for $5 \leq q \leq 15$, have a unique graceful labeling (short of symmetries and complementarity). This makes us suspect that the graceful labeling given by Proposition 3 is unique for $C_9 + I_q$, $q \geq 5$, and makes us wonder if there are any other cone graphs with unique graceful labelings.

An interesting problem, as a possible approach to the Graceful Tree Conjecture, is to decide if a graceful graph is 0-rotatable [4, 17]. For cone graphs, it suffices to check if there is a graceful labeling in which the vertex label 0 (or $m$) is in the independent set. So far, the only non-0-rotatable graceful cone graphs we know of satisfy $p \equiv 1 \pmod 4$.

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