# Immediate versus Eventual Conversion: Comparing Geodetic and Hull Numbers in $P_3$-Convexity

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**Abstract.** We study the graphs $G$ for which the hull number $h(G)$ and the geodetic number $g(G)$ with respect to $P_3$-convexity coincide. These two parameters correspond to the minimum cardinality of a set $U$ of vertices of $G$ such that the simple expansion process that iteratively adds to $U$, all vertices outside of $U$ that have two neighbors in $U$, produces the whole vertex set of $G$ either eventually or after one iteration, respectively. We establish numerous structural properties of the graphs $G$ with $h(G) = g(G)$, which allow the constructive characterization as well as the efficient recognition of all triangle-free such graphs. Furthermore, we characterize the graphs $G$ that satisfy $h(H) = g(H)$ for every induced subgraph $H$ of $G$ in terms of forbidden induced subgraphs.

**Keywords.** Hull number; geodetic number; $P_3$-convexity; irreversible 2-threshold processes; triangle-free graphs

## 1 Introduction

As one of the most elementary models of the spreading a property within a network — like sharing an idea or disseminating a virus — one can consider a graph $G$, a set $U$ of vertices of $G$ that initially possess the property, and an iterative process whereby new vertices $u$ enter $U$ whenever sufficiently many neighbors of $u$ are already in $U$. The simplest choice for "sufficiently many" that results in interesting effects is 2. This choice leads to the *irreversible* 2-*threshold processes* considered by Dreyer and Roberts [5]. Similar models were studied in various contexts such as statistical physics, social networks, marketing, and distributed computing under different names such as bootstrap percolation, influence dynamics, local majority processes, irreversible dynamic monopolies, catastrophic fault patterns and many others [1–3, 5, 7–10].

From the point of view of discrete convexity, the above spreading process is nothing but the formation of the convex hull of the set $U$ of vertices of $G$ with respect to the so-called $P_3$-*convexity* in $G$, where a set $C$ of vertices of $G$ is considered to be $P_3$*-convex* if no vertex of $G$ outside of $C$ has two neighbors in $C$. A $P_3$*-hull set* of $G$ is a set of vertices whose $P_3$-convex hull equals the whole vertex set of $G$, and the minimum cardinality of a $P_3$-hull set of $G$ is the $P_3$*-hull number* $h(G)$ of $G$.

Closely related to the notion of hull sets and the hull number of a graph are geodetic sets and the geodetic number. A $P_3$*-geodetic set* of graph $G$ is a set of vertices such that every vertex $u$ of $G$ either belongs to the set or has two neighbors in the set. The minimum cardinality of a $P_3$-geodetic set of $G$ is the $P_3$*-geodetic number* $g(G)$ of $G$. Different types of graph convexities have been considered in the literature, and the definitions of hull and geodetic sets change accordingly. For the special case of $P_3$-convexity, the $P_3$-geodetic number coincides with the well-studied 2-domination number [6].

In view of the iterative spreading process considered above, a hull set *eventually* distributes the property within the entire network, whereas a geodetic set spreads the property within the entire network *in exactly one iteration.* In [11] we considered spreading processes with arbitrary deadlines between 1 and $\infty$. Clearly, every geodetic set is a hull set, which implies

$$h(G) \leq g(G) \tag{1}$$

for every graph $G$. Furthermore, both parameters are computationally hard in general, and efficient algorithms are only known for quite restricted graph classes [4, 6].

In the present paper we study graphs that satisfy (1) with equality. After summarizing useful notation and terminology, we collect numerous structural properties of such graphs in Section 2. Based on these properties, we construct a large subclass of those graphs in Section 3, comprising all triangle-free such graphs. In Section 4 we derive an efficient algorithm for the recognition of the triangle-free graphs that satisfy (1) with equality. In Section 5 we give a complete characterization in terms of forbidden induced subgraphs of the class of all graphs for which (1) holds with equality for every induced subgraph. Finally, we conclude with some open problems in Section 6.

### 1.1 Notation and Terminology

We consider finite and simple graphs and digraphs, and use standard terminology. For a graph $G$, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. For a vertex $u$ of a graph $G$, the neighborhood of $u$ in $G$ is denoted $N_G(u)$ and the degree of $u$ in $G$ is denoted $d_G(u)$. A vertex of a graph whose removal increases the number of components is a cut vertex. A set $C$ of vertices of $G$ is $P_3$*-convex* exactly if no vertex of $G$ outside $C$ has two neighbors in $C$. The $P_3$*-convexity of* $G$ is the collection $\mathcal{C}(G)$ of all $P_3$-convex sets. Since we only consider $P_3$-convexity, we will omit the prefix "$P_3$-" from now on.

For a set $U$ of $G$, let the *interval* $I_G(U)$ *of* $U$ *in* $G$ be the set $U \cup \{u \in V(G) \setminus U \mid |N_G(u) \cap U| \geq 2\}$, and let $H_G(U)$ denote the convex hull of $U$ in $G$, that is, $H_G(U)$ is the unique smallest set in $\mathcal{C}(G)$ containing $U$. Within this notation, $U$ is a geodetic set of $G$ if $I_G(U) = V(G)$, and $U$ is a hull set of $G$ if $H_G(U) = V(G)$. The inequality (1) follows from the immediate observation that $I_G(U) \subseteq H_G(U)$ for every set $U$ of vertices of some graph $G$.

If $U$ is a hull set of $G$, then there is an acyclic orientation $D$ of a spanning subgraph of $G$ such that the in-degree $d_D^-(u)$ is 0 for every vertex $u$ in $U$ and 2 for every vertex $u$ in $V(G) \setminus U$. We call $D$ a *hull proof for $U$ in $G$*.

Since the hull number and the geodetic number are both additive with respect to the components of $G$, we consider the set of graphs

$$\mathcal{H} = \{G \mid G \text{ is a connected graph with } h(G) = g(G)\}.$$

## 2 Structural properties of graphs in $\mathcal{H}$

We collect some structural properties of the graphs in $\mathcal{H}$ in the form of lemmas which will be required to prove our main results in the next section. The proofs of many lemmas in this section, however, were omitted due to space limitations and left to an extended version of this paper.

Let $G$ be a fixed graph in $\mathcal{H}$. Let $W$ be a geodetic set of $G$ of minimum order and let $B = V(G) \setminus W$. By definition, every vertex in $B$ has at least two neighbors in $W$. Therefore, $G$ has a spanning bipartite subgraph $G_0$ with bipartition $V(G_0) = W \cup B$ such that every vertex in $B$ has degree exactly 2 in $G_0$. Let $E_1$ denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in the same component of $G_0$ and let $E_2$ denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in distinct components of $G_0$. Note that, by construction, $W$ is a geodetic set of $G_0$. Since $|W| = g(G) = h(G) \leq h(G_0) \leq g(G_0) \leq |W|$, we obtain $h(G_0) = g(G_0) = |W|$, that is, $G_0$ has no geodetic set and no hull set of order less than $|W|$. Thus, if $C$ is a component of $G_0$, then $W \cap V(C)$ is a minimum geodetic set of $C$ as well as a minimum hull set of $C$.

**Lemma 1.** *Let $C$ be a component of $G_0$.*

*(i) No two vertices in $C$ are incident with edges in $E_2$.*
*(ii) If some vertex $u$ in $C$ is incident with at least two edges in $E_2$, then $u$ belongs to $B$ and $u$ is a cut vertex of $C$.*

*Proof.* (i) We consider different cases. If two vertices $w$ and $w'$ in $V(C) \cap W$ are incident with edges in $E_2$, then let $P : w_1 b_1 \ldots w_{l-1} b_{l-1} w_l$ be a shortest path in $C$ between $w = w_1$ and $w' = w_l$. The set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction.

If a vertex $w$ in $V(C) \cap W$ and a vertex $b$ in $V(C) \cap B$ are incident with edges in $E_2$, then let $P : w_1 b_1 \ldots w_l b_l$ be a shortest path in $C$ between $w = w_1$ and $b = b_l$. Note that $b$ has a neighbor in $G_0$ that does not belong to $P$. Therefore, the set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction.

Finally, if two vertices $b$ and $b'$ in $V(C) \cap B$ are incident with edges in $E_2$, then let $P : b_1 w_1 \ldots b_{l-1} w_{l-1} b_l$ be a shortest path in $C$ between $b = b_1$ and $b' = b_l$. Note that $b$ and $b'$ both have neighbors in $G_0$ that do not belong to $P$.

Therefore, the set $(W \setminus \{w_1, \ldots, w_{l-1}\}) \cup \{b_2, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction.

(ii) If a vertex $w$ in $V(C) \cap W$ is incident with at least two edges in $E_2$, then $W \setminus \{w\}$ is a hull set of $G$, which is a contradiction.

If a vertex $b$ in $V(C) \cap B$ that is not a cut vertex of $C$ is incident with at least two edges in $E_2$, then let $P : w_1b_1 \ldots w_{l-1}b_{l-1}w_l$ be a path in $C$ avoiding $b$ between the two neighbors $w_1$ and $w_l$ of $b$ in $G_0$. The set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction and completes the proof.

**Lemma 2.** *If $G_0$ is not connected, no two vertices in $W$ that belong to the same component of $G_0$ are adjacent.*

*Proof.* For contradiction, we assume that $ww'$ is an edge of $G$ where $w$ and $w'$ are vertices in $W$ that belong to the same component $C$ of $G_0$. Since $G$ is connected, there is an edge $uv$ in $E_2$ with $u \in V(C)$ and $v \in V(G) \setminus V(C)$.

First, we assume that $u$ belongs to $W$. Let $P : w_1b_1 \ldots w_{l-1}b_{l-1}w_l$ be a shortest path in $C$ between $u = w_1$ and a vertex $w_l$ in $\{w, w'\}$. Note that $l = 1$ is possible. The set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction.

Next, we assume that $u$ belongs to $B$. Let $P : b_1w_1 \ldots b_lw_l$ be a shortest path in $C$ between $u = b_1$ and a vertex $w_l$ in $\{w, w'\}$. Note that $l = 1$ is possible. Furthermore, note that $b_1$ has a neighbor in $G_0$ that does not belong to $P$. The set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_2, \ldots, b_l\}$ is a hull set of $G$, which is a contradiction and completes the proof.

**Lemma 3.** *If $G_0$ is not connected and $C$ is a component of $G_0$, then there are no two vertices $w$ in $V(C) \cap W$ and $b$ in $V(C) \cap B$ such that $wb \in E(G) \setminus E(G_0)$.*

*Proof.* For contradiction, we assume that $wb$ is an edge of $G$ where $w$ in $W$ and $b$ in $B$ belong to the same component $C$ of $G_0$. Since $G$ is connected, there is an edge $uv$ in $E_2$ with $u \in V(C)$ and $v \in V(G) \setminus V(C)$.

First, we assume that $u \in W$. Let $P$ be a shortest path in $C$ between $u$ and a vertex $u'$ in $\{w, b\}$. If $u' = w$, then let $P : w_1b_1 \ldots b_{l-1}w_l$ where $u = w_1$ and $w = w_l$. Note that $l = 1$ is possible. In this case the set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction. If $u' = b$, then let $P : w_1b_1 \ldots b_{l-1}w_lb_l$ where $u = w_1$ and $b = b_l$. Note that $l = 1$ is possible. Furthermore, note that $b$ has a neighbor in $G_0$ that does not belong to $P$. In this case, the set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_1, \ldots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction.

Next, we assume that $u = b$. Let $P : b_1w_1 \ldots b_lw_l$ be a shortest path in $C$ between $b = b_1$ and $w = w_l$. Note that the edge $bw$ does not belong to $C$, hence $l \geq 2$. Furthermore, note that $b$ has a neighbor in $G_0$ that does not belong to $P$. In this case, the set $(W \setminus \{w_1, \ldots, w_l\}) \cup \{b_2, \ldots, b_l\}$ is a hull set of $G$, which is a contradiction.

Finally, we assume that $u \in B \setminus \{b\}$. Let $P$ be a shortest path in $C$ between $u$ and a vertex $u'$ in $\{w, b\}$. If $u' = w$, then let $P : b_1w_1 \ldots b_lw_l$, where $u = b_1$

and $w = w_l$. Note that $l = 1$ is possible. Furthermore, note that $w$ is the unique neighbor of $b$ in $P$, and that $u$ has a neighbor in $G_0$ that does not belong to $P$. In this case, the set $(W \setminus \{w_1, \dots, w_l\}) \cup \{b_2, \dots, b_l\}$ is a hull set of $G$, which is a contradiction. If $u' = b$, then let $P : b_1 w_1 \dots w_{l-1} b_l$, where $u = b_1$ and $b = b_l$. In this case, the set $(W \setminus \{w_1, \dots, w_{l-1}\}) \cup \{b_2, \dots, b_{l-1}\}$ is a hull set of $G$, which is a contradiction and completes the proof.

**Lemma 4.** *Let $G_0$ be disconnected and let $b$ and $b'$ be two vertices in $B$ that belong to the same component $C$ of $G_0$ satisfying $bb' \in E_1$.*

*(i) Neither $b$ nor $b'$ is incident with an edge in $E_2$.*
*(ii) If some vertex $w$ in $V(C) \cap W$ is incident with an edge in $E_2$ and $P : w_1 b_1 \dots w_l b_l$ is a path in $C$ between $w = w_1$ and a vertex $b_l$ in $\{b, b'\}$, then $w_l$ is adjacent to both $b$ and $b'$, and $C$ contains no path between $b$ and $b'$ that does not contain $w_l$.*
*(iii) If some vertex $b''$ in $(V(C) \cap B) \setminus \{b, b'\}$ is incident with an edge in $E_2$ and $P : b_1 w_1 \dots w_{l-1} b_l$ is a path in $C$ between $b'' = b_1$ and a vertex $b_l$ in $\{b, b'\}$, then $w_{l-1}$ is adjacent to both $b$ and $b'$ and $C$ contains no path between $b$ and $b'$ that does not contain $w_{l-1}$.*

**Lemma 5.** *If $C$ is a component of $G_0$, then there are no two vertices $w$ and $w'$ of $C$ that belong to $W$ and two edges $e$ and $e'$ that belong to $E(G) \setminus E(G_0)$ such that $w$ is incident with $e$, $w'$ is incident with $e'$, and $e'$ is distinct from $ww'$.*

**Lemma 6.** *If $C$ is a component of $G_0$, then there are no two edges $wb$ and $wb'$ that belong to $E(G) \setminus E(G_0)$ with $w \in W \cap V(C)$ and $b, b' \in B \cap V(C)$.*

**Lemma 7.** *If $G_0$ is connected and $G$ is triangle-free, then there are no two edges $ww'$ and $bb'$ in $G$ with $w, w' \in W$ and $b, b' \in B$.*

**Lemma 8.** *If $G_0$ is connected and $G$ is triangle-free, then there are no two edges $wb$ and $b'b''$ in $G$ with $w \in W$ and $b, b', b'' \in B$.*

**Lemma 9.** *If $G_0$ is connected and $G$ is triangle-free, then there are no two distinct edges $bb'$ and $b''b'''$ in $G$ with $b, b', b'', b''' \in B$.*

## 3 Constructing all triangle-free graphs in $\mathcal{H}$

Let $\mathcal{G}_0$ denote the set of all bipartite graphs $G_0$ with a fixed bipartition $V(G_0) = B \cup W$ such that every vertex in $B$ has degree exactly 2.

We consider four distinct operations that can be applied to a graph $G_0$ from $\mathcal{G}_0$.

- **Operation $\mathcal{O}_1$**
  Add one arbitrary edge to $G_0$.
- **Operation $\mathcal{O}_1'$**
  Select two vertices $w_1$ and $w_2$ from $W$ and arbitrarily add new edges between vertices in $\{w_1, w_2\} \cup (N_{G_0}(w_1) \cap N_{G_0}(w_2))$.

– **Operation $\mathcal{O}_2$**
  Add one arbitrary edge between vertices in distinct components of $G_0$.
– **Operation $\mathcal{O}_3$**
  Choose a non-empty subset $X$ of $B$ such that all vertices in $X$ are cut vertices of $G_0$ and no two vertices in $X$ lie in the same component of $G_0$. Add arbitrary edges between vertices in $X$ so that $X$ induces a connected subgraph of the resulting graph. For every component $C$ of $G_0$ that does not contain a vertex from $X$, add one arbitrary edge between a vertex in $C$ and a vertex in $X$.

Let $\mathcal{G}_1$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_1$ once to a connected graph $G_0$ in $\mathcal{G}_0$. Let $\mathcal{G}_1'$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_1'$ once to a connected graph $G_0$ in $\mathcal{G}_0$. Let $\mathcal{G}_2$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_2$ once to a graph $G_0$ in $\mathcal{G}_0$ that has exactly two components. Let $\mathcal{G}_3$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_3$ once to a graph $G_0$ in $\mathcal{G}_0$ that has at least three components. Note that $\mathcal{O}_3$ can only be applied if $G_0$ has at least one cut vertex that belongs to $B$.

Finally, let

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_1' \cup \mathcal{G}_2 \cup \mathcal{G}_3. \tag{2}$$

Since the operation $\mathcal{O}_1'$ allows that no edges are added, the set $\mathcal{G}_1'$ contains all connected graphs in $\mathcal{G}_0$.

**Theorem 1.** *$\mathcal{G} \subseteq \mathcal{H}$.*

*Proof.* Let $G$ be a graph in $\mathcal{G}$ that is obtained by applying some operation to a graph $G_0$ in $\mathcal{G}_0$. Let $V(G_0) = B \cup W$ be the fixed bipartition of $G_0$. Since every vertex in $B$ has two neighbors in $W$, the partite set $W$ is a geodetic set of $G$ and therefore $g(G) \leq |W|$. By (1), it suffices to show that $h(G) \geq |W|$ to conclude the proof. For contradiction, we assume that $U$ is a hull set of $G$ with $|U| < |W|$. Let $D$ be a hull proof for $U$ in $G$.

The proof naturally splits into four cases according to which of the four sets $\mathcal{G}_1$, $\mathcal{G}_1'$, $\mathcal{G}_2$, and $\mathcal{G}_3$ the graph $G$ belongs to. Due to space limitation, we give the details of the proof only for one case.

**Case 1** $G \in \mathcal{G}_1$.

Let $W_1 = W \setminus U$ and $B_0 = B \cap U$. Note that, by the above assumption, $|W_1| > |B_0| \geq 0$.

We claim that there is at most one vertex $w$ in $W_1$ for which the set $N_{G_0}(w)$ contains exactly one vertex of $B_0$ and that for every other vertex $w'$ in $W_1$, the set $N_{G_0}(w')$ contains at least two vertices of $B_0$. In other words, there is a vertex $w^*$ in $W_1$ such that

$$|N_{G_0}(w) \cap B_0| \geq \begin{cases} 1, w = w^*, \\ 2, w \in W_1 \setminus \{w^*\}. \end{cases} \tag{3}$$

Let $w$ be a vertex in $W_1$. Since $|W_1| > |B_0|$, we may assume that $N_{G_0}(w)$ contains at most one vertex from $B_0$. Let $x$ and $y$ denote the two in-neighbors of $w$ in $D$. Let $e$ denote the edge added by operation $\mathcal{O}_1$.

If $x$ belongs to $W$, then $e$ is the edge $xw$. Hence $y \in B$ and $d_G(y) = 2$. Therefore $y \in B_0$, that is, $y \in N_{G_0}(w) \cap B_0$. Furthermore, for every other vertex $w'$ in $W_1 \setminus \{w\}$, its two in-neighbors $x'$ and $y'$ in $D$ both belong to $B$ and are not incident with $e$. Hence $d_G(x') = d_G(y') = 2$ and therefore $x', y' \in B_0$, that is, $x', y' \in N_{G_0}(w) \cap B_0$. Hence, we may assume that $x$ and $y$ both belong to $B$.

If $e$ is the edge $wx$, then we obtain as above that $y \in N_{G_0}(w) \cap B_0$. Hence $x$ does not belong to $B_0$. This implies that the two edges of $G_0$ incident with $x$ are both oriented towards $x$ in $D$. For every other vertex $w'$ in $W_1 \setminus \{w\}$, it follows that its two in-neighbors $x'$ and $y'$ in $D$ satisfy $x', y' \in N_{G_0}(w) \cap B_0$. Hence, we may assume that $e$ is neither $wx$ nor $wy$.

Since $N_{G_0}(w)$ contains at most one element from $B_0$, we may assume that $e$ is incident with $x$ and oriented towards $x$ in $D$. This implies that $y \in N_{G_0}(w) \cap B_0$. Furthermore, for every other vertex $w'$ in $W_1 \setminus \{w\}$, its two in-neighbors $x'$ and $y'$ in $D$ both belong to $B$, and, if they are incident with $e$, then $e$ is not oriented towards them in $D$. This implies that $x'$ and $y'$ belong to $B_0$, that is, $x', y' \in N_{G_0}(w') \cap B_0$.

Altogether, the existence of a vertex $w^*$ in $W_1$ with (3) follows. If $m$ denotes the number of edges in $G_0$ between $W_1$ and $B_0$, then (3) implies $m \geq 2(|W_1| - 1) + 1$. Furthermore, every vertex in $B$ has degree 2 in $G_0$ and therefore $m \leq 2|B_0|$. Thus, $2|W_1| - 1 \leq 2|B_0|$. Since both cardinalities are integers, we obtain $|W_1| \leq |B_0|$, hence

$$|U| = |W \cap U| + |B \cap U| \geq |W \cap U| + |W \setminus U| = |W|,$$

which is a contradiction. This completes the proof.

In conjunction, the results in Sections 2 and Theorem 1 allow for a complete constructive characterization of the triangle-free graphs in $\mathcal{H}$.

**Corollary 1.** *If $\mathcal{T}$ denotes the set of all triangle-free graphs, then $\mathcal{G} \cap \mathcal{T} = \mathcal{H} \cap \mathcal{T}$.*

*Proof.* Theorem 1 implies $\mathcal{G} \cap \mathcal{T} \subseteq \mathcal{H} \cap \mathcal{T}$. For the converse inclusion, let $G$ be a triangle-free graph in $\mathcal{H}$. Similarly as in Section 2, let $W$ be a minimum geodetic set of $G$, let $B = V(G) \setminus W$, and let $G_0$ be a spanning bipartite subgraph of $G$ with bipartition $V(G_0) = W \cup B$ such that every vertex in $B$ has degree exactly 2 in $G_0$. Let $E_1$ denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in the same component of $G_0$ and let $E_2$ denote the set of edges in $E(G) \setminus E(G_0)$ between vertices in distinct components of $G_0$.

First, we assume that $G_0$ is connected. In this case, $E_1 = E(G) \setminus E(G_0)$. For contradiction, we assume that $E_1$ contains two edges $e$ and $e'$. By Lemmas 5 and 6, the edges $e$ and $e'$ are not both incident with vertices in $W$. We may therefore assume that $e$ connects two vertices from $B$. Now, since $G$ is triangle-free, Lemmas 7, 8, and 9 imply a contradiction. Hence $E_1$ contains at most one edge, which implies $G \in \mathcal{G}_1 \cup \mathcal{G}_1'$.

Next, we assume that $G_0$ is disconnected. By Lemmas 2 and 3, all vertices incident with edges in $E_1$ belong to $B$. For contradiction, we assume that $E_1$ is not empty. Let $bb' \in E_1$, where $b$ and $b'$ belong to some component $C$ of $G_0$. Since $G$ is connected but $G_0$ is not, some vertex of $C$ is incident with an edge $f$ from $E_2$. By Lemma 4, the edge $f$ is not incident with $b$ or $b'$. Furthermore, by Lemma 4 (ii) and (iii), $G$ necessarily contains a triangle, which is a contradiction. Hence $E_1$ is empty. Now Lemma 1 immediately implies $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, which completes the proof.

Corollary 1 implies several restrictions on the cycle structure of a triangle-free graph $G$ in $\mathcal{H}$. Let $G_0$ with bipartition $B \cup W$ be the underlying graph in $\mathcal{G}_0$. Clearly, all cycles of $G$ that are also cycles of $G_0$ are of even length and alternate between $B$ and $W$. Furthermore, at most one of the vertices from $B$ in such a cycle can have degree more than 2 in $G$. If $G_0$ is connected, the cycles of $G$ are either such cycles of $G_0$ or they contain the unique edges in $E(G) \setminus E(G_0)$. If $G_0$ has two components, then $G$ arises from $G_0$ by adding a bridge and all cycles of $G$ are also cycles of $G_0$. Finally, if $G_0$ has at least three components and $X$ is as described in $\mathcal{O}_3$, then $X$ induces an arbitrary connected triangle-free graph in $G$, that is, the cycle structure of $G[X]$ can be quite complicated. Nevertheless, all cycles in $G[X]$ contain only vertices of degree at least 4 in $G$. All further cycles of $G$ are totally contained within one component of $G_0$ and contain at least one vertex from $B$ that has degree 2 in $G$.

## 4 Recognizing all triangle-free graphs in $\mathcal{H}$

By Corollary 1, the structure of the triangle-free graphs in $\mathcal{H}$ is quite restricted. In fact, it is not difficult to recognize these graphs in polynomial time. This section is devoted to the details of a corresponding algorithm.

Let $G$ be a given connected triangle-free input graph. By Corollary 1, the graph $G$ belongs to $\mathcal{H}$ if and only if either $G$ belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ or $G$ belongs to $\mathcal{G}_3$.

**Lemma 10.** *It can be checked in polynomial time whether $G \in \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$.*

*Proof.* By definition, the graph $G$ belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ if and only if deleting at most one edge from $G$ results in a graph in $\mathcal{G}_0$ with at most two components. Since the graphs in $\mathcal{G}_0$ can obviously be recognized in linear time, it suffices to check whether $G \in \mathcal{G}_0$ and to consider each edge $e$ of $G$ in turn and check whether $G - e \in \mathcal{G}_0$. Since the graphs in $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ have a linear number of edges, all this can be done in quadratic time. This completes the proof.

In view of Lemma 10, we may assume from now on that $G$ does not belong $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. The following lemma is an immediate consequence of the definition of operation $\mathcal{O}_3$.

**Lemma 11.** *If $G$ belongs to $\mathcal{G}_3$, then there is a vertex $x$ of $G$ of degree at least three and two edges $e_l = xy_l$ and $e_r = xy_r$ of $G$ incident with $x$ such that, in the* *graph $G'$ that arises by deleting from $G$ all edges incident with $x$ except for $e_l$ and $e_r$, the component $C(x, e_l, e_r)$ of $G'$ that contains $x$ has the following properties:*

*(i)* *$x$ is a cut vertex of $C(x, e_l, e_r)$;*
*(ii)* *$C(x, e_l, e_r)$ has a unique bipartition with partite sets $B_l \cup \{x\} \cup B_r$ and $W_l \cup W_r$;*
*(iii)* *Every vertex in $B_l \cup \{x\} \cup B_r$ has degree 2 in $C(x, e_l, e_r)$;*
*(iv)* *$B_l \cup W_l$ and $B_r \cup W_r$ are the vertex sets of the two components of $C(x, e_l, e_r) - x$ such that $y_l \in W_l$ and $y_r \in W_r$;*
*(v)* *None of the deleted edges connects $x$ to a vertex from $V(C(x, e_l, e_r)) \setminus \{x\}$;*
*(vi)* *$W_l$ and $W_r$ both contain a vertex of odd degree.*

*Proof.* Choosing as $x$ one of the vertices from the non-empty set $X$ in the definition of $\mathcal{O}_3$ and choosing as $e_1$ and $e_2$ the two edges of $G_0$ incident with $x$, the properties (i) to (v) follow immediately. Note that $C(x, e_l, e_r)$ is the component of $G_0$ that contains $x$. For property (vi), observe that the number of edges of $C(x, e_l, e_r)$ between $B_l \cup \{x\}$ and $W_l$ is exactly $2|B_l| + 1$, that is, it is an odd number, which implies that not all vertices of $W_l$ can be of even degree. A similar argument applies to $W_r$. This completes the proof.

The key observation for the completion of the algorithm is the following lemma, which states that the properties from Lemma 11 uniquely characterize the elements of $X$.

**Lemma 12.** *If $G$ belongs to $\mathcal{G}_3$ and a vertex $x$ of $G$ of degree at least three and two edges $e_l = xy_l$ and $e_r = xy_r$ of $G$ incident with $x$ are such that properties (i) to (vi) from Lemma 11 hold, then*

*(i)* *$G$ is obtained by applying operation $\mathcal{O}_3$ to a graph $G_0$ in $\mathcal{G}_0$ with at least three components such that $x$ belongs to the set $X$ used by operation $\mathcal{O}_3$ and*
*(ii)* *$C(x, e_l, e_r)$ defined as in Lemma 11 is the component of $G_0$ that contains $x$.*

We proceed to the main result in this section.

**Theorem 2.** *For a given triangle-free graph $G$, it can be checked in polynomial time whether $h(G) = g(G)$ holds.*

*Proof.* Clearly, we can consider each component of $G$ separately and may therefore assume that $G$ is connected. Let $n$ denote the order of $G$. By Lemma 10, we can check in $O(n^2)$ time whether $G$ belongs $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. If this is the case, then Corollary 1 implies $h(G) = g(G)$. Hence, we may assume that $G$ does not belong to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. Note that there are $O(n^3)$ choices for a vertex $x$ of $G$ and two incident edges $e_l$ and $e_r$ of $G$. Furthermore, note that for every individual choice of the triple $(x, e_l, e_r)$, the properties (i) to (vi) from Lemma 11 can be checked in $O(n)$ time. Therefore, by Lemmas 11 and 12, in $O(n^4)$ time, we can

– either determine that no choice of $(x, e_l, e_r)$ satisfies the conclusion of Lemma 11, which, by Corollary 1, implies $h(G) \neq g(G)$,
– or find a suitable triple $(x, e_l, e_r)$ and reduce the instance $G$ to a smaller instance $G^- = G - V(C(x, e_l, e_r))$.

Since the order of $G^-$ is at least three less than $n$, this leads to an overall running time of $O(n^5)$. This completes the proof.

## 5 Forbidden induced subgraphs

It is an easy exercise to prove $h(G) = g(G)$ whenever $G$ is a path, a cycle, or a star.

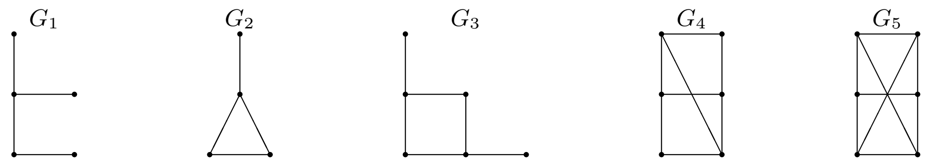


**Fig. 1.** The five graphs $G_1, \dots, G_5$.

**Theorem 3.** *If $G$ is a graph, then $h(H) = g(H)$ for every induced subgraph $H$ of $G$ if and only if $G$ is $\{G_1, \dots, G_5\}$-free.*

*Proof.* Since $3 = h(G_1) = h(G_3) < g(G_1) = g(G_3) = 4$ and $2 = h(G_2) = h(G_4) = h(G_5) < g(G_2) = g(G_4) = g(G_5) = 3$, the "only if"-part of the statement follows. In order to prove the "if"-part, we may assume, for contradiction, that $G$ is a connected $\{G_1, \dots, G_5\}$-free graph with $h(G) < g(G)$. We consider different cases.

**Case 1** *$G$ contains a triangle $T : abca$.*

Since $G$ is $G_2$-free, no vertex has exactly one neighbor on $T$.

If some vertex has no neighbor on $T$, then, by symmetry, we may assume that there are two vertices $u$ and $v$ of $G$ such that $uva$ is a path and $u$ has no neighbor on $T$. Since $G$ is $G_2$-free, we may assume that $v$ is adjacent to $b$. Now $u$, $v$, $a$, and $b$ induce $G_2$, which is a contradiction. Hence every vertex has at least one neighbor on $T$. This implies that the vertex set of $G$ can be partitioned as

$$V(G) = \{a, b, c\} \cup N(\{a, b\}) \cup N(\{a, c\}) \cup N(\{b, c\}) \cup N(\{a, b, c\}),$$

where $N(S) = \{u \in V(G) \setminus \{a, b, c\} \mid N_G(u) \cap \{a, b, c\} = S\}$.

If two vertices, say $u$ and $v$, in $N(\{a, b\})$ are adjacent, then $u$, $v$, $a$, and $c$ induce $G_2$, which is a contradiction. Hence, by symmetry, each of the three sets $N(\{a, b\})$, $N(\{a, c\})$, and $N(\{b, c\})$ is independent. If some vertex $u$ in $N(\{a, b\})$

is not adjacent to some vertex $v$ in $N(\{a,c\})$, then $u$, $v$, $a$, and $b$ induce $G_2$, which is a contradiction. Hence, by symmetry, there are all possible edges between every two of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$. If some vertex $u$ in $N(\{a,b\})$ is not adjacent to some vertex $v$ in $N(\{a,b,c\})$, then $u$, $v$, $a$, and $c$ induce $G_2$, which is a contradiction. Hence, by symmetry, there are all possible edges between the two sets $N(\{a,b\}) \cup N(\{a,c\}) \cup N(\{b,c\})$ and $N(\{a,b,c\})$. If $N(\{a,b\})$ contains exactly one vertex, say $u$, then $I_G(\{u,c\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$. Hence, by symmetry, none of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$ contains exactly one vertex. If there are two vertices in $N(\{a,b\})$, say $u_1$ and $u_2$, and two vertices in $N(\{b,c\})$, say $v_1$ and $v_2$, then $u_1$, $u_2$, $v_1$, $v_2$, $a$, and $c$ induce $G_5$, which is a contradiction. Hence no two of the three sets $N(\{a,b\})$, $N(\{a,c\})$, and $N(\{b,c\})$ contain at least two vertices.

Altogether, we may assume, by symmetry, that $N(\{a,c\})$ and $N(\{b,c\})$ are empty. Now $I_G(\{a,b\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$ and completes the proof in this case.

**Case 2** *$G$ contains no triangle but a cycle of length four $C : abcda$.*

If some vertex has no neighbor on $C$, then, by symmetry, we may assume that there are two vertices $u$ and $v$ of $G$ such that $uva$ is a path. Since $G$ is triangle-free, $v$ is not adjacent to $b$ or $d$. Hence $u$, $v$, $a$, $b$, and $d$ induce $G_1$, which is a contradiction. Hence every vertex has at least one neighbor on $C$. Since $G$ is triangle-free, this implies that the vertex set of $G$ can be partitioned as

$$V(G) = \{a,b,c,d\} \cup N(\{a\}) \cup N(\{b\}) \cup N(\{c\}) \cup N(\{d\}) \cup N(\{a,c\}) \cup N(\{b,d\}),$$

where $N(S) = \{u \in V(G) \setminus \{a,b,c,d\} \mid N_G(u) \cap \{a,b,c,d\} = S\}$.

If there is a vertex $u$ in $N(\{a\})$ and a vertex $v$ in $N(\{c\})$, then $u$ and $v$ are adjacent, because $G$ is $G_3$-free. Now $u$, $v$, $a$, $b$, and $d$ induce $G_1$, which is a contradiction. Hence, by symmetry, we may assume that $N(\{c\}) \cup N(\{d\})$ is empty. If there is a vertex $u$ in $N(\{b\})$ and a vertex $v$ in $N(\{a,c\})$, then $u$ and $v$ are not adjacent, because $G$ is $G_4$-free. Now $u$, $v$, $a$, $b$, and $d$ induce $G_1$, which is a contradiction. Hence, by symmetry, one of the two sets $N(\{b\})$ and $N(\{a,c\})$ is empty and one of the two sets $N(\{a\})$ and $N(\{b,d\})$ is empty. If there is a vertex $u$ in $N(\{a,c\})$ and a vertex $v$ in $N(\{b,d\})$, then $u$, $v$, $a$, $b$, $c$, and $d$ induce either $G_4$ or $G_5$, which is a contradiction. Hence, by symmetry, we may assume that $N(\{b,d\})$ is empty.

Since $G$ is $G_1$-free, there are all possible edges between the two sets $N(\{a\})$ and $N(\{b\})$.

Since $G$ is $G_1$-free, both of the sets $N(\{a\})$ and $N(\{b\})$ contain at most one vertex.

Since $G$ is $G_1$-free, there is no edge between $N(\{a\})$ and $N(\{a,c\})$.

If both of the sets $N(\{a\})$ and $N(\{b\})$ are not empty, then $G$ is a graph of order 6 with $h(G) = g(G) = 3$, which is a contradiction. Hence, by symmetry, we may assume that

$$V(G) = \{a,b,c,d\} \cup N(\{a\}) \cup N(\{a,c\}).$$

If $N(\{a\})$ is empty, then $I_G(\{a,c\}) = V(G)$, which implies the contradiction $2 \leq h(G) \leq g(G) \leq 2$. Hence $N(\{a\})$ contains exactly one vertex, say $u$, and $I_G(\{a,c,u\}) = V(G)$, which implies $g(G) \leq 3$. If $H_G(U) = V(G)$ for some set $U$ of vertices of $G$, then $u \in U$. In view of the structure of $G$, it follows easily that $h(G) \geq 3$, which implies the contradiction $h(G) = g(G)$ and completes the proof in this case.

**Case 3** *$G$ does not contain a triangle or a cycle of length four.*

If $G$ contains no vertex of degree at least 3, then $G$ is a path or a cycle, which implies the contradiction $h(G) = g(G)$. Hence, we may assume $G$ contains a vertex of degree at least 3. Since $G$ is $G_1$-free, $G$ is a star, which implies the contradiction $h(G) = g(G)$ and completes the proof.

## 6 Conclusion

Several open problems/tasks are immediate.

- Give a constructive characterization of all graphs in $\mathcal{H}$.
- Describe an efficient algorithm to recognize all graphs in $\mathcal{H}$.

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