Polynomial time algorithm for the Radon number of grids in the geodetic convexity

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Abstract

The Radon number of a graph is the minimum integer r such that all sets of at least r vertices of the graph can be partitioned into two subsets whose convex hulls intersect. We present a near-linear $O(d \log d)$ time algorithm to calculate the Radon number of d-dimensional grids in the geodetic convexity. To date, no polynomial time algorithm was known for this problem.

1 Introduction

The concept of convexity in graphs was borrowed from its well-known geometric counterpart, where a subset S of the Euclidean space is convex if and only if, for every two points $x, y \in S$, the *interval* of x, y, consisting of the straight segment connecting x to y, is entirely contained in S. Formally, a *convexity space* consists of a pair $(V, \mathcal{C})$, where V is a set — the *ground set* — and $\mathcal{C}$ is a collection of subsets of V — the *convex sets* — such that $\mathcal{C}$ contains both V and the empty set, and $\mathcal{C}$ is closed under arbitrary intersections.

Several types of graph convexities have been considered in the literature, with applications ranging from statistical physics and distributed computing to marketing and social networks. In the *geodetic* convexity, a subset S of the vertices of a graph G is convex if and only if, for every two vertices $x, y \in S$, all vertices in a shortest path between x and y in G also belong to S.

Given a subset V' of the ground set V of some convexity space, the *convex hull* of V', denoted $H(V')$, is the smallest convex subset of V containing V'. Nearly one hundred years ago, Johann Radon formulated a celebrated theorem stating that every every set with at least $d+2$ points in R^d can be partitioned into two subsets whose convex hulls intersect [5]. A natural question concerns what happens when we consider some general ground set V instead of R^d. The *Radon number* of V is defined as the minimum integer r such that every subset of V with at least r elements can be partitioned in two sets whose convex hulls intersect, and it has been used to model a number of problems that occur, for instance, in social networks.

A simple reduction from the maximum clique problem can be used to prove the NP-hardness of finding the Radon number of a graph in the geodetic convexity, hence a natural task is to determine such parameter for particular graph classes. We are interested in the Cartesian products of d paths of arbitrary sizes (i.e. d-dimensional grids). While a lot of insight on the problem was gained in [1], determining the exact Radon number of such graphs in polynomial time was still outstanding. We introduce an algorithm that achieves that in near-linear $O(d \log d)$ time.

2 The basics

If R is a subset of vertices of graph G, then a partition $R = R_1 \cup R_2$ is a *Radon partition* if $H(R_1) \cap H(R_2) \neq \emptyset$. A set with no Radon partitions is an *anti-Radon set*. The Radon number of a graph G is therefore the size of the maximum anti-Radon set of G plus one.

A grid $G = \text{Grid}(n_1, \ldots, n_d)$ is the Cartesian product of d paths $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_d}$. If R is a subset of the vertices of a grid, then $H(R)$ equals the Cartesian product of the 1-dimensional convex hulls of the d projections of R onto the different dimensions ρ_j, for $j \in [d] := \{1, \ldots, d\}$.

Having observed that, one can check whether a partition $R = R_1 \cup R_2$ is a Radon partition by inspecting the projections of R onto each dimension. If, for some $j \in [d]$, the greatest (smallest) coordinate of R_1 on ρ_j is less (greater) than the smallest (greatest) coordinate of R_2 on ρ_j, then $H(R_1) \cap H(R_2) = \emptyset$. In this case, we say the projections onto ρ_j of R_1 appear *all-to-the-left* (*all-to-*

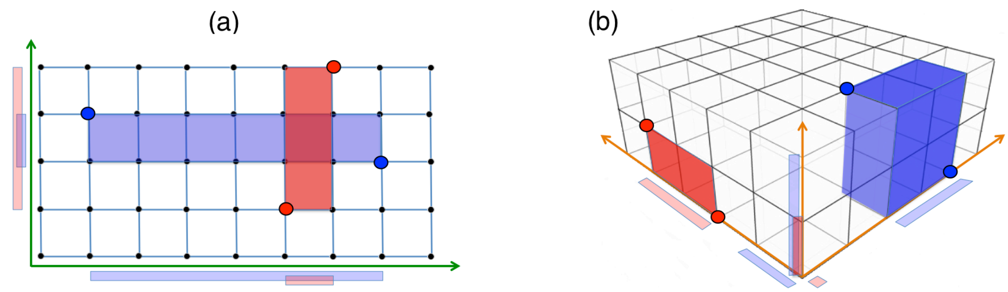

Fig. 1. (a) a Radon partition: on all dimensions the convex hulls of the projections intersect; (b) *not* a Radon partition: on two dimensions, the projections of one of the partite sets are all-to-the-left of the projections of the other partite set.

the-right) of the projections of R_2. Figure 1 illustrates the idea.

In [2], Eckhoff determined the Radon number of the convexity space defined on $\mathbb{R}^d$ by the Manhattan metric $(u, v) \mapsto ||u - v||_1$ as

$$r(d) := \min\left\{ r \in \mathbb{N} : \binom{r}{\lfloor \frac{r}{2} \rfloor} > 2d \right\}. \tag{1}$$

In [3], Jamison-Waldner observed that Eckhoff's result could be instantly leveraged to $\mathrm{Grid}(n_1, \ldots, n_d)$ provided $n_j \geq r(d) - 1$, for all $j \in [d]$. However, if the grid dimensions are not as large, Eckhoff's result gives but an upper bound. In the next section, we will be able to obtain the geodetic Radon number of grids exactly.

3 The algorithm

Let $G = \mathrm{Grid}(n_1, \ldots, n_d)$. The algorithm we propose evaluates the existence of anti-Radon sets of decreasing size r, starting from Jamison-Waldner's upper bound $r = r(d) - 1$, until it finds one.

Note that the actual coordinates of the vertices of a given set do not really matter for the sake of deciding whether or not such set is an anti-Radon set. What does matter is solely the *permutation* of their projections onto each dimension.

If a given grid G admits an anti-Radon set R of size r, then, for all possible partitions $R = R_1 \cup R_2$ (referred to as Radon partition *candidates*), there must be a dimension ρ_j of G where the projections of R_1 appear all-to-the-left of the projections of R_2, or vice-versa. We say ρ_j *eliminates* that candidate. The idea of the algorithm is to subsequently attempt to remove each candidate (from a set containing initially all partitions of $[r]$) by assigning a dimension that eliminates it.

A Radon partition candidate $R = R_1 \cup R_2$, with $k = |R_1| \leq |R_2|$, $k \in [\lfloor r/2 \rfloor]$, is referred to as a *$k$-candidate* for set R.

Claim 3.1 *Given a set R with r elements and an integer $k \in [\lfloor r/2 \rfloor]$, the number $n(r,k)$ of k-candidates for set R is equal to $\binom{r}{k}$, if $k \neq r/2$, and is equal to $\frac{1}{2}\binom{r}{k}$, if $k = r/2$.*

Proof. Trivial. Note only that, if r is even, then for $k = r/2$, both complementary partite sets will have the same size, so the number of such k-candidates is only half the number of subsets of R with size $k = r/2$. □

Claim 3.2 *For all k, the maximum number $s(r,k)$ of k-candidates that can be eliminated by any given dimension is 2, if $k \neq r/2$, and is 1, if $k = r/2$.*

Proof. The claim is again immediate for all $k \neq r/2$. If $k = r/2$, though, then a sequence of projections onto ρ_j that presents exactly k projections all-to-the-left (of the remaining projections) must also leave exactly $r-k = k$ projections all-to-the-right. Since those two sets are complementary, they belong to the same partition of R. □

The *potential* of dimension ρ_j stands for the maximum number of candidates that can be eliminated by dimension ρ_j. Along the execution of the algorithm, such value gets decremented by one unit every time a new candidate is eliminated by ρ_j. The following claim is actually a slight rephrasing of Corollary 4 presented in [1], therefore we omit its proof. It settles the initial potential of each dimension.

Claim 3.3 *Given a subset R comprising r vertices of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ and some $j \in \{1, \ldots, d\}$, the maximum number $t(r,j)$ of partitions $R = R_1 \cup R_2$ that can be eliminated by dimension ρ_j equals $\min\{n_j, r\} - 1$.*

Theorem 3.4 *The algorithm described in Figure 2 calculates the geodetic Radon number of a given d-dimensional grid in $O(d \log d)$ time.*

Proof. If the algorithm fails to find an anti-Radon set with size r, then, for some k, the algorithm was not able to eliminate one k-candidate out of the $n(r,k)$ total given by Claim 3.1. But that can only happen after the algorithm has tried to eliminate the maximum possible number $s(r,k)$ of such candidates per dimension (according to Claim 3.2). And now we have two possible scenarios:

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algorithm: geodetic_radon_number_of_grids
input: the dimension sizes n_j of a grid G, for j = 1, ..., d
output: the Radon number of G
1   for r = r(d) − 1, r(d) − 2, ..., 1 do
2       for j = 1, ..., d do
3           potential[j] ← t(r, j) as in Claim 3.3
4       for k = ⌊r/2⌋, ⌊r/2⌋ − 1, ..., 1 do
5           n ← n(r, k) as in Claim 3.1
6           s ← s(r, k) as in Claim 3.2
7           for j = 1, ..., d do
8               count_eliminated[j, k] ← 0
9           while n > 0 do
10              find j* such that potential[j*] is maximum, satisfying
                    (i) potential[j*] > 0, and
                    (ii) count_eliminated[j, k] < s
11              if no such j* exists, continue with the next r in line 1
12              n −= 1
13              potential[j*] −= 1
14              count_eliminated[j*, k] += 1
15      return r + 1
16  return 1
```

Fig. 2. Algorithm that calculates the Radon number of a grid.

- The grid does not admit any anti-Radon set of size r, in which case the answer given by the procedure is certainly correct; or
- The grid does admit an anti-Radon set R of size r. For $j \in [d]$, let $z(k, j)$ denote the number of k-candidates $R^* = R_1^* \cup R_2^*$ eliminated by ρ_j. Let also $\mathcal{Z}(k, j)$ be the collection of such candidates. Since the algorithm failed to eliminate a k-candidate, for some k, then there is one dimension ρ_j whose potential was less than $z(k, j)$ by the time the algorithm was considering that failed candidate. But this means that, for some $k' > k$ and some $j' \neq j$, there is a k'-candidate in $\mathcal{Z}(k', j')$ which was associated by the algorithm to the j-th dimension instead of the j'-th, i.e. which led the algorithm to decrement the potential of ρ_j instead of $\rho_{j'}$. Even after further eliminations that may have decreased the potential of $\rho_{j'}$ before our k-candidate was considered, there must be a dimension $\rho_{j''}$ whose potential is at least $z(k, j'') + 1$ by the time the algorithm considers subsets of size k. In other words, the unit of potential that was not discounted from $\rho_{j'}$ cannot

have vanished. If $z(k, j'') < 2$, then the extra unit in the potential of $\rho_{j''}$ could have been used to eliminate our failed k-candidate. Since that was obviously not the case (otherwise our k-candidate would *not* have failed to be eliminated), then $z(k, j'') = 2$, and the potential of $\rho_{j''}$ is at least $z(k, j'') + 1 = 3$ by the time the algorithm considers the elimination of our k-candidate — and at all times before that, particularly by the time the algorithm eliminated the last candidate that reduced the potential of ρ_j to something less than $z(k, j)$. Hence the potential of ρ_j was at most $z(k, j) \leq s(k, j) \leq 2$ just prior to that particular elimination. And this clearly contradicts the algorithm's choice of the dimension whose potential will be decremented, for it always chooses the dimension with the greatest potential (Figure 2, line 10), and the potential of $\rho_{j''}$ was at least 3 at that precise moment.

Now, if the algorithm returns $r + 1$, then it succeeded in eliminating all Radon partition candidates for sets with size r. The existence of an anti-Radon set with size r can be guaranteed based on the construction discussed in the proof of Theorem 6 in [1].

As for the execution time, the proposed algorithm runs $O(r(d))$ iterations in the worst case, each one taking $O(2^{O(r(d))})$ time. Thus, we can rely on the nice approximation

$$\binom{r}{\lfloor \frac{r}{2} \rfloor} \approx \frac{2^r}{\sqrt{r+1}} \cdot \sqrt{\frac{2}{\pi}}$$

for the central binomial coefficient [4] to derive $r(d) = O(\log d)$, and the overall time of our algorithm is $O(d \log d)$. $\square$

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